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THE ELEMENTS OF PHYSICS

VOL. III

LIGHT AND SOUND

The
Co.

THE

ELEMENTS OF PHYSICS

A COLLEGE TEXT-BOOK

BY
EDWARD L. NICHOLS
AND
WILLIAM S. FRANKLIN

IN THREE VOLUMES

VOL. III.

LIGHT AND SOUND

NEW EDITION, REVISED AND REWRITTEN

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PREFACE.

This edition of Nichols and Franklin's Elements of Physics, Vol. III., is entirely rewritten, a chapter dealing chiefly with Architectural Acoustics is added and a carefully arranged set of problems is given.

The authors' thanks are due to Professor P. A. Lambert, Dr. C. C. Schenck, Dr. G. W. Stewart, Mr. Barry MacNutt, and to Mr. C. M. Crawford for many valuable suggestions, and to Mr. C. M. Crawford for great assistance in reading proof.

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95736

TABLE OF CONTENTS.

CHAPTER I.
LIGHT AND SOUND DEFINED. VELOCITY OF TRANSMISSION.

CHAPTER II.
WAVES AND WAVE TRAINS. RAYS.

CHAPTER III.
REFLECTION AND REFRACTION. CONCAVE MIRRORS.

CHAPTER IV.
LENSES AND LENS SYSTEMS.

CHAPTER V.
SIMPLE OPTICAL INSTRUMENTS.

CHAPTER VI.
LENS IMPERFECTIONS AND THEIR COMPENSATION.

CHAPTER VII.
DISPERSION. SPECTRUM ANALYSIS.

CHAPTER VIII.
INTERFERENCE AND DIFFRACTION. THE DIFFRACTION GRATING.

CHAPTER IX.
PHOTOMETRY.

CHAPTER X.
COLOR. COLOR BLINDNESS.

CHAPTER XI.
POLARIZATION AND DOUBLE REFRACTION.

CHAPTER XII.
RADIANT HEAT.

CHAPTER XIII.
THE VIBRATION OF A PARTICLE. LOUDNESS, PITCH AND TIMBRE OF SOUNDS.

CHAPTER XIV.
THE FREE VIBRATIONS OF ELASTIC SYSTEMS.

CHAPTER XV.
IMPRESSED VIBRATIONS AND RESONANCE.

CHAPTER XVI.
THE EAR AND HEARING.

CHAPTER XVII.
MUSICAL INTERVALS AND SCALES. MUSICAL EXPRESSION.

CHAPTER XVIII.
ARCHITECTURAL ACOUSTICS. MISCELLANEOUS PHENOMENA.

APPENDIX A.
THE DIFFERENTIAL EQUATIONS OF WAVE MOTION.

PROBLEMS.

THE ELEMENTS OF PHYSICS.

VOLUME III.

CHAPTER I.

LIGHT AND SOUND DEFINED; VELOCITY.

1. Sensory nerves. — The sensory nerves of the human body lead from regions near the surface of the body to the central organs of the nervous system. The outer ends of these nerves are exposed in such a way as to be excited or set into commotion by physical disturbances in the region surrounding the body. This commotion is transmitted to the central organs producing commotion there, and we experience what we call a *sensation*. The physical disturbance which excites a nerve is called a *stimulus*.

2. Proper stimuli. *End organs; localization.* — Those physical disturbances to which a set of sensory nerves are especially sensitive are called the *proper stimuli* of that set of nerves. A set of sensory nerves is rendered especially sensitive to its proper stimuli by being provided with *specialized end organs* which are easily affected by those stimuli, and by being so *located* as to be very largely protected by the surrounding tissues of the body from other excitation.

3. Optic nerves. *Light, the sensation; light, the proper stimulus.*—The end organs (rods and cones) of the nerves of sight are situated in the retina of the eye. They are well protected by the surrounding bones from all physical disturbances, except such as can reach them through the transparent humors of the eye.

The *sensation* which is experienced when the optic nerves are excited is called *light*. That physical disturbance which constitutes the proper stimulus of the optic nerves is also called *light*.

Severe mechanical shocks frequently excite the optic nerves, producing sensations of light; and the electric current, when caused to flow through the eye, gives a sensation of light.

4. Auditory nerves. *Sound, the sensation; sound, the proper stimulus.* — The end organs of the nerves of hearing are located in the chambers of the inner ear, and they are well protected by the massive bones of the head from all disturbances except such as reach them in the following manner: Rapid variations of pressure on the ear-drum, due to *vibratory motion of the atmosphere*, set the ear-drum into vibration. These vibrations are transmitted across an empty cavity by a chain of tiny bones to a membrane which covers an opening (the *oval window*) to the chambers of the inner ear. A second opening (the *round window*) to the inner ear is covered by a membrane that is entirely free. The inner ear is filled with fluid. The vibration of the membrane which covers the oval window causes this fluid to surge back and forth between oval window and round window through the chambers of the inner ear, and the end organs of the auditory nerves which are submerged in this fluid are excited.

The *sensation* which is experienced when the auditory nerves are excited is called *sound*. That *physical disturbance* which constitutes the proper stimulus of the auditory nerves is also called *sound*.

5. Transmission of light and of sound. — We have come by experience to associate distant objects with our sensations of sight and hearing, and to draw from these sensations more or less complicated inferences concerning such distant objects. This process is called the *seeing* or the *hearing* of the distant objects.

The attempt to explain the connection between a distant object and the sensations of light and hearing which are associated with it, has led philosophers to assume the existence of physical agencies which are transmitted to us from distant objects and affect our organs of sight and hearing. The various theories of light and sound which have thus arisen are explained in the immediately following articles.

6. The corpuscular theory of light. — The phenomena of shadows and the obstruction of vision of a distant object by intervening objects, show that *light travels sensibly in straight lines.* In accordance with this fact it was the accepted theory, until long after the time of Sir Isaac Newton, that light consists of particles or corpuscles which are thrown off from luminous bodies at great velocity, traveling in straight lines until reflected (or stopped) by objects upon which they impinge. This was called the *corpuscular theory* of light.

7. The wave theory of light and of sound. — The most comprehensive understanding of the phenomena of light and sound has been reached on the hypothesis that light and sound are wave-like disturbances which pass out in all directions from luminous bodies and from sonorous bodies respectively. No attempt will be made to lead up to this hypothesis by preliminary discussion. Its justification, in the reader's mind, will become more and more complete as he has occasion to use it.

8. Transmitting media. *The luminiferous ether.* — The conception of light as a wave-like disturbance requires the assumption of a transmitting medium. The fact that light reaches us from the sun and stars across apparently empty space, and the fact that no known material substance is capable of transmitting waves with anything approaching the enormous velocity of light, necessitate the assumption of a special light-transmitting medium, the *ether*, which fills all space.

Sound, on the other hand, cannot travel through a vacuum. The transmission of sound is, in fact, accomplished by a wave-like disturbance of air, or water, or other material medium.

9. The elastic-solid theory of light. — In earlier days light waves were considered to be waves of distortion traveling through an elastic medium. Light waves are of a kind known as transverse waves and transverse waves can be propagated only in a medium which has *rigidity*, that is in a solid medium. This old wave theory of light is therefore called the *elastic-solid* theory.

A curious difficulty arose in this old theory, namely, that the transmitting medium, the ether, was necessarily assumed to have the properties of a solid, in order to transmit light waves and yet the planets traveled through this solid ether without perceptible friction. This hypothetical ether possessed the rigidity of steel and it was at the same time a fluid compared with which the thinnest air was as viscid as pitch!

10. The electromagnetic theory of light. — The theory of electricity and magnetism, as developed by Maxwell, is based upon the assumption of a medium capable of transmitting electric and magnetic forces; and, about 1870, Maxwell predicted that an electric or magnetic disturbance would send out waves in this medium and that the velocity of these waves, calculated from electromagnetic data, would be identically the velocity of light. The theory of electromagnetic waves was developed by Maxwell and proposed by him as a new basis for the wave theory of light, — the *electromagnetic theory of light.*

Most of the essential points in Maxwell's theory have been verified in recent years, especially by the experiments of Hertz on electromagnetic waves in 1887, and the electromagnetic theory of light is now universally accepted.

The difference, from the philosophical point of view, between the elastic-solid theory of light and the electromagnetic theory has been pointed out most clearly by Heaviside.* The electromagnetic theory is abstract and highly generalized, and it rests mainly on facts; the elastic-solid theory is concrete and highly specialized, and it rests mainly upon hypotheses.

The elastic-solid theory is very particular in saying what light consists of, and for this reason it is useful in giving clear ideas to one who is not practiced in thinking in terms of the abstract notions of electricity and magnetism. It is for this reason that the terminology of the elastic-solid theory is used in this text in the chapters on wave-motion and on polarization and double refraction.

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* "Electromagnetic Theory," Vol. I., p. 325.

11. The measurement of the velocity of sound.* — It is a familiar fact that sound requires a perceptible time to reach the ear from a sounding body. The first attempt to measure accurately the velocity of sound was made by a committee of members of the French Academy of Sciences in 1738. The observers were placed at the Paris Observatory and at three distant stations visible from the observatory. Every ten minutes a cannon was fired at one of these stations and the observers at the other stations noted the time intervals which elapsed between the flash and the sound of the cannon. The light flash is transmitted almost instantaneously so that the observed time intervals were taken to be the time intervals required for the sound to travel over the measured distances between the stations.

In 1822 this experiment was repeated at Paris in a slightly modified form. Two stations were selected, cannon were fired at these stations alternately at intervals of ten minutes, and the time intervals were observed as before. By firing at the two stations alternately the influence of the wind was eliminated. The distance between the stations was 18,622.27 meters. The observed time interval between flash and sound at one station was 54.84 seconds and at the other station it was 54.43 seconds. These data give 340.88 meters per second (331.2 meters per second at 0° C.).

In all these experiments it was assumed that the same length of time is required for an observer to become aware of the flash after the light falls on his retina as is required for him to become aware of the report of the gun after the sound enters his ear. In fact the time required in each case is considerable and the two times are probably not the same, especially inasmuch as in the case of the flash the observer would be taken by surprise, while the flash would serve as a warning and the observer would be prepared for a quick perception of the sound of the report.

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* For a more complete account of the researches described in this article, see Wüllner, "Experimentalphysik" (5th ed., Vol. I., p. 928).

The French physicist, Regnault, performed experiments on the velocity of sound in which the instant of discharge of the cannon and the instant of arrival of the sound were automatically recorded by a chronograph (see Vol. I.). The bullet from the gun cut a wire stretched across the muzzle, thus breaking an electrical circuit and actuating the recording pen of the chronograph ; at the receiving station the sound struck a light diaphragm, the motion of which closed an electrical circuit and actuated the recording pen of the same chronograph. Regnault's experiments gave for the velocity of sound 330.7 meters per second at 0° C.

The velocity of sound in air depends upon the temperature, as is explained in the appendix.

There are various indirect methods of measuring the velocity of sound. To these reference will be made in subsequent chapters. These methods are especially valuable in the determination of the velocity of sound in other gases than air, and in solids and liquids where it is not practicable to employ a path thousands or even hundreds of meters in length. The following table gives the velocity of sound in various substances in terms of the velocity of sound in air :

TABLE.

Velocities of Sound in Different Media.

Velocity of sound in air at 0° C., 331.8 meters per second.

SUBSTANCES.	VELOCITY OF SOUND COMPARED WITH THAT IN AIR.
Lead	4.257
Gold	6.424
Silver	8.057
Copper	11.167
Platinum	8.467
Iron	15.108
Glass	15.29
Water	4.3

12. The measurement of the velocity of light.* — A Danish astronomer, Roemer (1675), was the first to show that light has a finite velocity. He found that the observed time of revolution of the satellites of Jupiter varies with the position of the earth in its orbit. When the earth is moving towards Jupiter the observed time of revolution of a satellite is less than the true time of revolution, and when the earth is moving away from Jupiter the observed time is greater than the true time; the true time of revolution being the mean of all the observed times during one revolution of the earth in its orbit. The principle of the variation of the observed time of revolution of Jupiter's satellites is as follows: Suppose a light signal is flashed at equal intervals of time. A stationary observer sees these flashes separated by the true time interval between them. If, during the interval between two flashes, the observer moves towards or away from the source of the flashes, the interval between observed flashes will be less or more than the true interval. The amount of this difference will be equal to the time required for light to pass over the distance that the observer has moved.

A laboratory method for measuring the velocity of light was devised by Fizeau in 1849. This method was employed under more favorable conditions by Cornu in 1874. Another method was devised by Foucault in 1850. This method has been used by various observers, notably by Michelson in 1880.

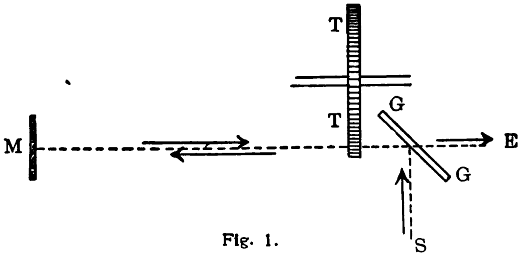

Fig. 1.

The method of Fizeau. — Light from a brilliant source *S*, Fig. 1, is reflected from an unsilvered glass *GG*, whence it passes between the cogs of a toothed wheel *TT* to a distant mirror *M*. From this mirror the light is reflected back so as to pass between the same pair of teeth on the wheel and a portion of this light passes on through the unsilvered glass *GG* to the observer's eye at *E*. If the wheel is now started rotating and slowly increased in speed, no light will reach the eye at *E* when the wheel has reached such a speed that, in the time, *t*, during which the light is traveling to the distant mirror and back again, the opening through which the outgoing light passed will have been supplanted by the first following tooth. As the speed is further increased light will again reach the eye at *E* to be again cut off when the wheel reaches such a speed that, during the time *t* the opening through which the outgoing light passed will have been supplanted by the second following tooth, and so on. In Cornu's experiments the distance from the wheel to the mirror *M* was 23 kilometers and the speed of the wheel was slowly increased until the light at *E* had disappeared and reappeared fifteen times. The wheel was then held steadily at this speed for a sufficient time for the speed to be accurately observed and from this speed the time *t* was determined as the time required for a given space to be supplanted by the fifteenth following tooth.

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* For full account of the researches upon the velocity of light, see Preston's "Light," Chapter XIX.

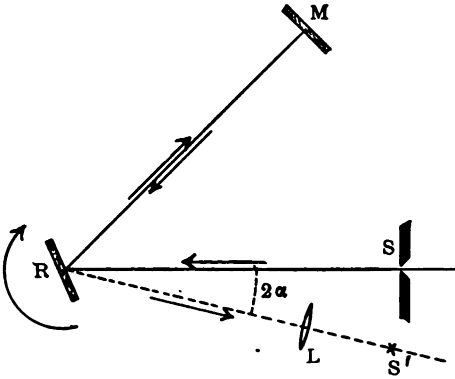

Fig. 2.

The method of Foucault. — Light from a brilliant source passes through a slit S, Fig. 2, and falls upon a rapidly rotating mirror R whence, once each revolution, the light is reflected towards a distant mirror M which reflects the light back to R. During the time t that the light is traveling from R to M and back again the mirror will have turned through an angle a and the returning ray will be reflected along the dotted line making an angle $2a$ with the incident ray. This returning ray passes through a lens L which forms an image of the slit at S'. The angle $2a$ is determined by measuring the distances SR and SS'. From this angle and the observed speed of the mirror the time t is found. In the experiments of Michelson the distance RM was about 600 meters and the mirror was driven at about 250 revolutions per second. Newcomb in 1882 carried out Foucault's method. He found, from an extended series of observations, a velocity of 299,778,000 meters per second in air (299,860,000 meters per second in vacuum). This is perhaps the most reliable determination that has been made.

CHAPTER II.

WAVES.

13. Nature of waves. — When a portion of an elastic medium is suddenly distorted and released, a wave of distortion passes out in all directions from that portion at a definite velocity. A distant portion of the medium is quiescent until this wave reaches it. It is distorted, and thrown into commotion, as the wave passes, after which it is again quiet. The movement along a stretched wire of a wave produced by a blow, and waves upon the surface of water, are familiar examples.

14. Longitudinal and transverse waves. — Fluids can transmit only those waves in which, as the wave passes, the particles of the medium move to and fro in *the direction of progression* of the wave. Such waves are called *longitudinal waves*. Thus waves in the air, sound waves, are longitudinal. On the other hand, solids can transmit longitudinal waves, and also waves in which the particles of the medium move to and fro in *a direction at right angles to the direction of progression* of the wave. Such waves are called *transverse waves*. Longitudinal and transverse waves have different velocities of progression in the same medium ; and, therefore, if a center of disturbance sends out waves of both kinds, one kind will outstrip the other and become isolated from it. A familiar example of this is afforded by the waves which pass along a tightly stretched wire which is struck sharply with a hammer. A person at the other end of the wire will hear a sharp click when the longitudinal wave reaches him, and another when the transverse wave reaches him. The air near the hammer will also be disturbed, and an air wave will reach the person, giving a third click between the other two.

Light waves are known to be transverse waves from the phenomena of polarization as explained in Chapter XI.

15. Waves from periodic disturbances; wave-trains.—A periodic disturbance is one which is repeated, in every detail, in equal intervals of time. The time interval τ during which one repetition of the disturbance takes place is called the *period* of the disturbance, and the number of repetitions per second is called the *frequency*. A periodic disturbance sends out what is called a *train of waves*. Each wave in a wave-train is exactly like its forerunner. The distance λ between similar parts of the adjacent waves of a train is called the *wave-length* of the wave-train; it is the distance traveled by the waves during the period τ. If the velocity of the waves be v, this distance is $v\tau$, so that

$$\lambda = v\tau. \tag{1}$$

16. Graphical representation of a wave-train.—As a wave-train passes by a given particle of a medium the particle oscillates to and fro, and the distance of the particle from its equilibrium position at a given instant is called the *displacement* of the particle at that instant. Consider a wave-train traveling through a me-

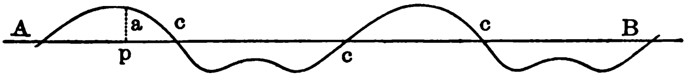

Fig. 3.

dium in the direction of the line AB, Fig. 3. At each point p of AB erect an ordinate a proportional to the displacement of the medium at p. If the wave-train is transverse the ordinate at each point p may be drawn upwards or downwards according as the displacement at p is upwards or downwards; if the wave-train is longitudinal the ordinate at each point p may be drawn *upwards* or *downwards* according as the displacement at p is *to the right* or *to the left* respectively. The curved line *ccc* constructed in this way represents the wave-train graphically. If this curve be imagined to move along at the velocity of the wave-train the actual motion of the various parts of the medium is clearly set forth.

17. Simple and compound wave-trains. — When a wave-train passes through a medium each particle of the medium oscillates. When each particle of the medium performs *simple harmonic motion* during the passage of a wave-train the wave-train is called a *simple wave-train.* A simple wave-train is represented graphically by a curve of sines. When, during the passage of a wave-train, the particles of a medium perform periodic movements which are not simply harmonic the wave-train is called a *compound wave-train.* A compound wave-train is represented graphically by a periodic curve (a curve of which the successive sections are exactly alike) which is not a curve of sines. Simple and compound wave-trains are discussed in Chapter XIII.

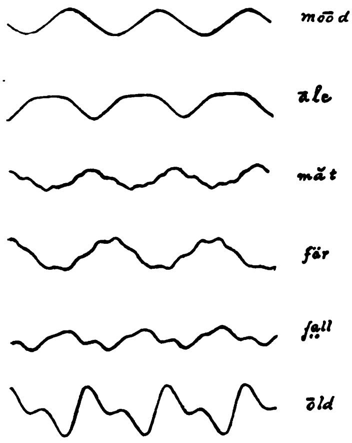

Fig. 4.

The curves of Fig. 4* represent the wave-trains which issue from the mouth when the indicated vowel sounds are produced by a baritone voice. The first curve represents a simple wave-train, the other curves represent compound wave-trains. It must not be thought from these examples that a given vowel sound is always associated with a wave-train of the same shape. This is by no means the case.

Equation of simple wave-train.—The equation of an ordinary curve of sines is :

$$y = B \sin \frac{2\pi x'}{\lambda} \qquad (a)$$

---

* By Professor L. B. Spinney. A very small mirror was mounted upon flexible hinges in a small rectangular opening in a wall. The vowel sound striking this wall caused the mirror to oscillate and a beam of light reflected from the mirror was focused upon a rapidly moving photographic plate. Fig. 4 is a reproduction full size of these photographic tracings.

in which y is the ordinate, x' is the abscissa, and λ is the wave-length. The curve represented by equation (a) is a stationary curve. To represent a curve of sines moving at velocity v in the direction of the x-axis, we must consider that after t seconds the curve will have moved a distance vt, so that a point of the curve of which the abscissa was x' at the start will, after t seconds, have an abscissa equal to $x' + vt$. Representing this abscissa by x, we have

$$x = x' + vt \qquad (b)$$

or

$$x' = x - vt \qquad (c)$$

Substituting this value of x' in equation (a) we have

$$y = B \sin 2\pi \left(\frac{x}{\lambda} - \frac{vt}{\lambda} \right)$$

or, writing τ for v/λ from equation (1) we have

$$= B \sin 2\pi \left(\frac{x}{\lambda} - \frac{t}{\tau} \right) \qquad (2)$$

18. Amplitude and phase. — The maximum displacement b, Fig. 5, of a simple wave-train is called the *amplitude* of the train.

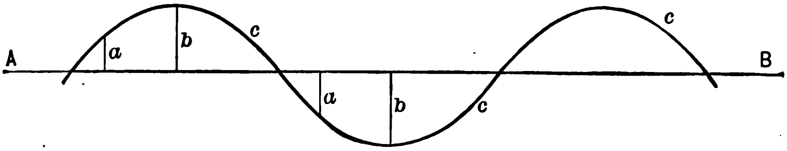

Fig. 5.

Two points in a simple wave-train distant a *whole wave-length* from each other always have equal displacements of the same sign and they are said to be *in the same phase;* two points in a simple wave-train distant *half a wave-length* from each other always have equal displacements of opposite sign and they are said to be *opposite in phase.* Thus the points *aa*, Fig. 5, are opposite in phase. The terms *crest* and *hollow*, as applied to water waves, are sometimes used to signify those points in a wave-train where the displacement has the greatest positive and the greatest negative value respectively.

19. The transmission of energy. *Intensity.* — When a disturbance is produced in an elastic medium all or nearly all of the energy of the disturbance is carried away by the waves. On the other hand waves give up their energy to obstacles which absorb them. Thus water waves give up nearly the whole of their energy when they strike a shelving shore where they are not reflected; light waves give up their energy to a black body. The energy which is transmitted by light waves and by sound waves is the physical measure of the intensity of the light or sound. Thus a beam of light of one square centimeter section would be said to have unit intensity if it transmitted one unit of energy per second. The energy transmitted by a wave-train is proportional to the square of its amplitude and inversely proportional to the square of its wave-length. This statement is worded to apply to sound waves and to light waves according to the elastic-solid theory.

20. The principle of superposition. — It is a familiar fact that a number of objects remain distinctly visible to a number of observers when the light, in passing from the various objects to the various observers, has to cross *the same region at the same time.* It is also true that, although a combination of sounds is more or less distracting to the attention, one sound does not sensibly alter the character of another which accompanies it. A number of waves can therefore traverse the same region simultaneously, each one independently of the others. *The actual displacement of a particle of the medium at a given instant is the vector sum of the displacements at that instant due to the separate waves.* Waves on the surface of water are, in the same way, independent of one another. An observer overlooking the sea can trace simultaneously the incoming ocean swell, the smaller waves caused by passing boats, the waves due to local wind, and the waves reflected from the shore.

21. Stationary wave-trains. — Consider two like wave-trains *AA* and *BB*, Figs. 6, 7, 8 and 9, moving in opposite directions as indicated by the heavy arrows. These wave-trains are supposed to be traversing the same region and *AA* is drawn above *BB* merely to avoid confusion. By the principle of superposition,

the actual displacement of each particle of the medium is equal to the sum of the displacements of that particle due to each wave-train and the actual velocity of each particle is equal to the sum of the velocities of that particle due to each wave-train. Now, the ordinates *O* of the wave-train *AA*, Fig. 6, which reach the points *ppp* as the wave-train *AA* moves to the right are at each instant equal and opposite to the ordinates *O'* of the wave-train *BB* which reach the points *ppp* as the wave-train *BB* moves to the left. Therefore the points *ppp* of the medium remain stationary. The regions between the points *pp*, on the other hand, move up and down (to right and left in case of a longitudinal wave) as the two wave-trains *AA* and *BB* travel through or over each other. The stationary points *ppp* are called *nodes*.*

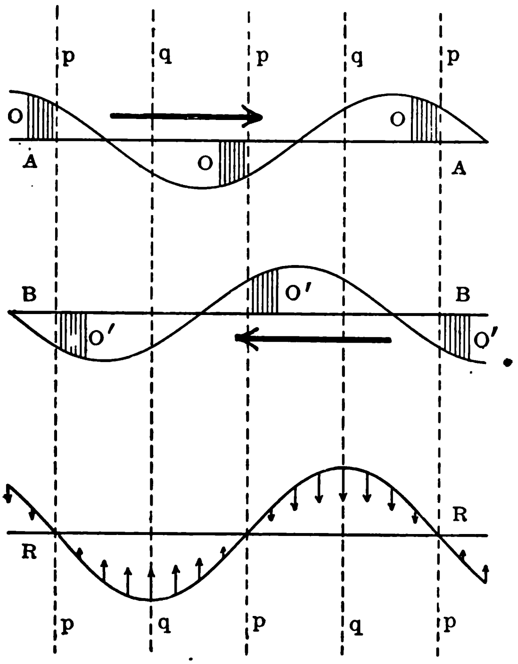

Fig. 6.

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* In an ordinary advancing wave-train no portion of the medium remains stationary, in fact every particle of the medium moves in the same way and to the same extent,

The intervening vibrating portions are called *vibrating segments.* The middle points *qq* of the vibrating segments are called *antinodes.* The resultant of the two wave-trains *AA* and *BB* is called a *stationary wave-train* inasmuch as this resultant has no progressive character.

Fig. 6 shows the positions of the two wave-trains *AA* and *BB* at a given instant and their resultant *RR*. The small vertical arrows show the velocities of the various parts of the medium.

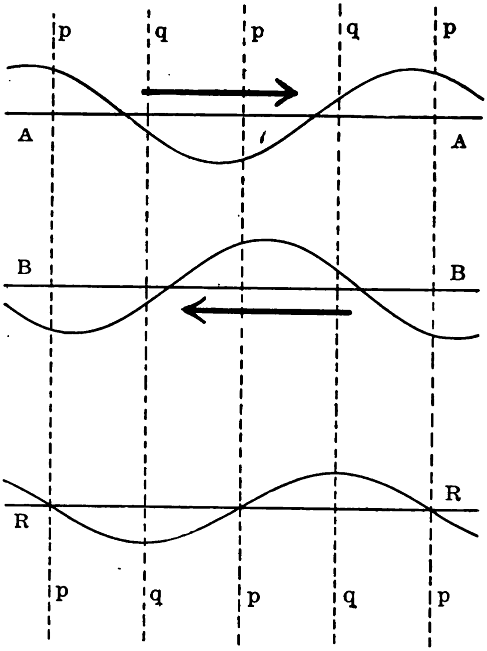

Fig. 7.

Fig. 7 shows the positions of the two oppositely moving wave-trains *AA* and *BB* and their resultant *RR* at a later instant when *AA* has moved $\frac{1}{16}$ of a wave-length to the right and *BB* has moved $\frac{1}{16}$ of a wave-length to the left.

but not simultaneously; each succeeding particle is a little later in its movements. In a stationary wave-train the medium is stationary at the nodes, the amplitude of motion increases from the node to the antinode, and all particles move simultaneously. That is, all particles in a vibrating segment move up and down together.

Fig. 8 shows the positions of *AA* and *BB* and their resultant *RR* (a straight line) at a still later instant when *AA* has moved $\frac{1}{8}$ of a wave-length to the right and *BB* has moved $\frac{1}{8}$ of a wave-length to the left. The small vertical arrows show the velocities of the various parts of the medium.

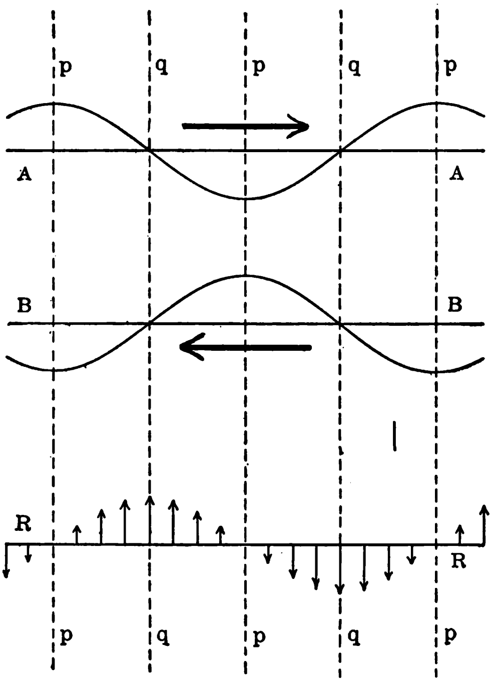

Fig. 8.

Fig. 9 shows the positions of *AA* and *BB* and their resultant *RR* at a still later instant when *AA* has moved $\frac{3}{16}$ of a wave-length to the right and *BB* has moved $\frac{3}{16}$ of a wave-length to the left.

The nodes of the stationary wave-train are the points where the two component trains are always *opposite in phase*, and the antinodes are the points where the two component trains are always *in the same phase*.

The portions of the medium at the antinodes of a stationary wave-train have at times considerable velocity, but are never distorted.

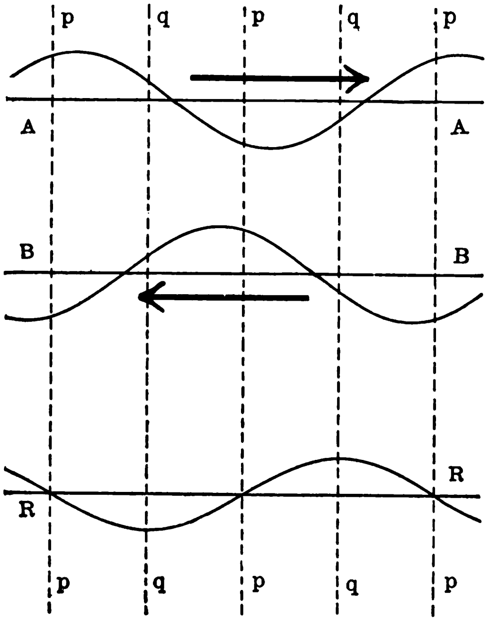

Fig. 9.

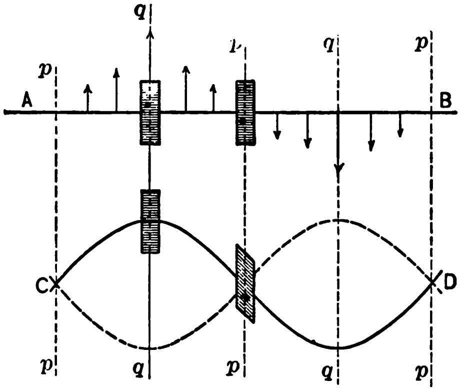

Fig. 10.

The portions at the nodes never move, but are at times much distorted. This, for transverse waves, is shown in Fig. 10. The shaded areas represent portions of the medium. *AB* represents the state of affairs when the velocities in the vibrating segments are greatest, as indicated by the arrows, and when the displacements are everywhere zero. *CD* represents the state of affairs one quarter of a period later, when the velocities are everywhere zero and the displacements are at their greatest. The shaded area at the node is distorted as shown.

In a stationary train of longitudinal waves the medium on the two sides of a node moves towards the node simultaneously and away from the node simultaneously and the medium at the node is alternately condensed and rarefied.

Remark. — Stationary wave-trains may result from the superposition of wave-trains of any shape, provided only that the advancing train is exactly similar to the receding one turned end for end and upside down.

22. Stationary wave-trains by reflection. — When a wave-train reaches the boundary of a medium, it is reflected. The reflected train is similar to the incident train, and if the incident train strikes the boundary normally, then a stationary train is produced by the superposition of the two.

If the boundary is *free*, that is, if the space outside of the boundary is empty space, then the surface layers of the medium are entirely free to move but the medium in the surface layers cannot be distorted since there is no possibility of these portions of the medium being squeezed, as it were, by the material behind them. In such a case there must be an antinode of the stationary wave-train at the reflecting boundary.

If the boundary is *fixed*, as, for example, where a very light medium such as air is bounded by a rigid wall, then the layers of the medium next the wall cannot move. In this case there is a node of the stationary wave-train at the reflecting boundary.

The component wave-trains of a stationary train are opposite in phase at a node, and in phase at an antinode, so that a reflected wave-train is *in phase* with the incident train at a free boundary, and *opposite* in phase at a rigid boundary. Reflection of the first kind is called *reflection without change of phase*, and reflection of the second kind is called *reflection with change of phase*.

An example of reflection without change of phase is furnished by the reflection of sound waves from below at the free surface of the water in a pond. The space above the water being filled with air acts very much as if it were empty space, the surface layer of water is free to move but it cannot be perceptibly compressed against the freely moving air above, and the sound waves are reflected without change of phase.

Stationary wave-trains may be easily produced by means of a stretched cord or wire, or with a stretched rubber tube. One end of the rubber tube is tied to a rigid support, and the other end is held in the hand. A series of periodic movements of the hand generates a wave-train on the tube. This train is reflected from the rigid end with change of phase ; and the tube is broken up into segments with intervening nodes as the reflected train and advancing train come into superposition. The rigid end of the tube is a node.

A stationary wave-train may likewise be produced in the air within a long glass tube. In this experiment both ends of the tube are open. The lips, applied to one end of the tube as to a bugle, are thrown into periodic motion producing a wave-train, which is reflected from the open end without change of phase. A stationary wave-train is produced in the tube when the reflected train and advancing train come into superposition. The open end of the tube is the center (nearly) of a vibrating segment. If lycopodium powder is strewn along the inside of the tube, it is swept into the nodal points by the violent to-and-fro motion of the air, giving a striking indication of the existence of the stationary train.

Remark. — In the case of light, the transmitting medium, the ether, permeates everything and fills all space, and the notions of free and rigid boundaries are scarcely legitimate although these notions are used in the elastic-solid theory of light. According to this theory there is a reflection with change of phase when light in air is reflected from the surface of glass, and a reflection without change of phase when light in glass is reflected from the surface which separates the glass from air. For a statement of this matter in terms of the electromagnetic theory of light see Vol. II., Chapter XV., Articles 236 and 237.

23. Wave front. — Consider a region, *AB* (Fig. 11), which is disturbed by a wave coming from a distant center of disturbance *C*. All parts of any small plane layer *AB* of the medium perpendicular to *r* will be similarly distorted, and the layer will move as a whole up and down or to and fro as the wave passes. *A surface so drawn as to pass through those parts of a wave where the distortion is everywhere the same, or those parts where the displacement is everywhere the same, is called a wave front.*

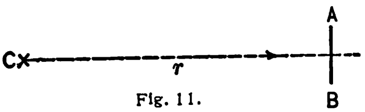

Fig. 11.

When a wave has come from a large number of centers of disturbance near at hand, it has in general, no definite front. Such a wave, may, however, be looked upon as a superposition of elementary waves coming each from a center of disturbance, and these elementary waves have definite fronts. A wave having a definite front may lose this character upon striking an obstacle.

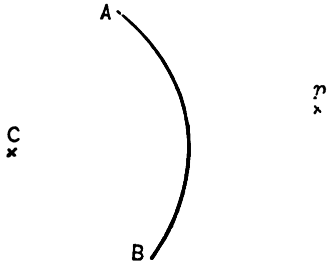

Fig. 12.

The direction of progression of a wave in an isotropic medium is at right angles to its front.*

A wave which has a plane wave front is called a *plane wave*. A wave which has a spherical wave front is called a *spherical* wave.

23. Huygens' principle. — Let *AB* (Fig. 12) be the instantaneous position of a wave which has come from a disturbance at *C*. The disturbance produced later at p as the wave passes that point is, of course, to be thought of as having come originally from C; it may, however, be considered to have come from the disturbance which constitutes the wave AB. In this latter case *each point of the wave AB is to be considered as a center of disturbance from which a spherical wave emanates.* The waves which thus emanate from each point of a *primary* wave are called *secondary* waves or *wavelets.* The actual disturbance produced at p is the resultant of the effects of all these wavelets.

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* The extraordinary wave in a doubly refracting crystal generally does not progress in a direction at right angles to its front.

25. Huygens' construction for wave front. — Let AB, Fig. 13, be the front of a wave advancing toward $A'B'$. Let it be required to find the wave front after a time has elapsed, during which the wave has traveled a distance r. Describe circles (spheres) of radius r from each point of the wave front AB. The *envelope* $A'B'$ of these circles (spheres) is the required wave front. These spheres are the secondary wavelets described in the previous article.

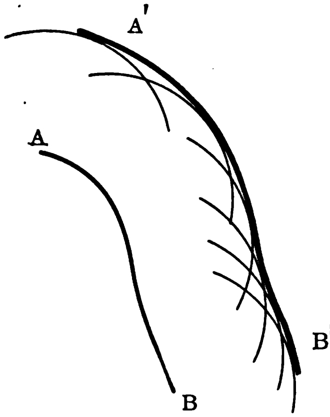

Fig. 13.

Each of the secondary wavelets emanates from an infinitesimal portion of AB, and represents an infinitesimal disturbance. The number of these secondary wavelets which work together to produce disturbance in a portion of their enveloping surface $A'B'$ is infinitely greater than the number which work together to produce a disturbance elsewhere. Therefore the disturbance along $A'B'$ is infinitely greater than elsewhere. This is equivalent to saying that the disturbance elsewhere is zero.

26. The ray of light. — Consider a train of waves WW, Fig. 14, passing out from a center of disturbance O. These waves progress at each point in a direction at right angles to the wave-front, and the lines *rrr* along which the wave-disturbance travels are called *rays*. Rays drawn from every point of a small portion of a wave front constitute a *pencil of rays*.

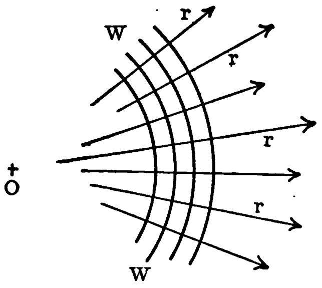

Fig. 14.

The conception of the ray carries with it the idea that wave disturbance is propagated in straight lines, that a wave-disturbance will not, for example, bend around an obstacle into the region behind the obstacle. In the case of water waves this is not the case, and in the case of sound waves we know by common experience that a sound can be heard around a corner. Light, on the other hand, seems ordinarily to travel in straight lines. In fact, however, light bends around a corner in the same way that sound does, but to an extent that is, under ordinary conditions, scarcely noticeable. This matter will be taken up again in the chapter on diffraction.

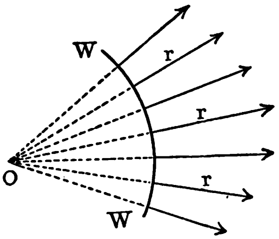

Fig. 15.

27. Homocentric pencil of rays.— When a portion of a wave front is a sector of a spherical surface (the entire wave front may not be a sphere) the rays from the portion intersect at the center of the sphere, the pencil of rays is said to be *homocentric*, and the center of the sphere is called the *focal point* of the pencil. The circle *WW*, Fig. 15, represents a spherical wave emanating from a center of disturbance *O*. In this case the wave is shown as traveling towards its convex side and the pencil of rays *rrr* is said to be *divergent*. In Fig. 16 *WW* represents a spherical wave which has

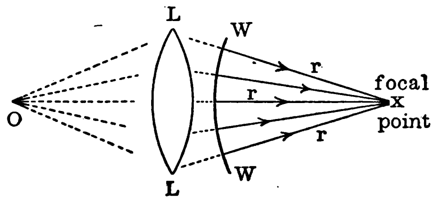

Fig. 16.

come from a center of disturbance *O* and has passed through a lens *LL*. In this case the wave *WW* is shown as traveling towards its concave side and the pencil of rays *rrr* is said to be *convergent*.

28. Astigmatic pencil of rays. — An astigmatic pencil of rays is a pencil that has no focal point. Consider a short arc of a circle, *AA*, Fig. 17, with its center at *C*. Imagine this circular arc to be turned slightly about a line *DD* as an axis. The circular arc will then describe a surface of which the principal sections are shown in Figs. 17 and 18, and *a bundle of straight lines drawn normally to this surface will intersect first along the line CC and then along the line DD.*

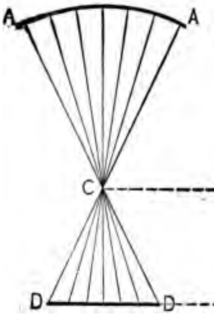

Fig. 17.

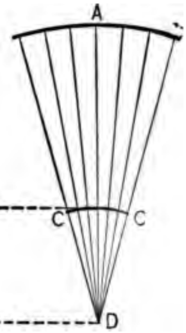

Fig. 18.

When a small portion of a wave front is shaped like the surface described by the arc *AA*, when turned about the line *DD*, Fig. 17, the pencil of rays from the portion of the wave front will pass first through the line *CC* and then through the line *DD*. Such a pencil of rays is called an *astigmatic pencil*. The lines *CC* and *DD* are called the *focal lines* of the pencil.

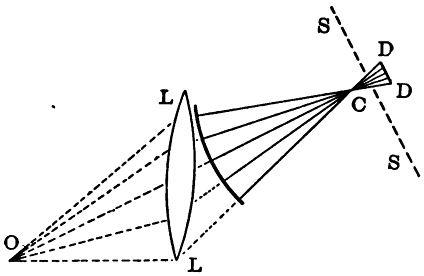

Fig. 19.

Example.—Light waves from a point *O*, Fig. 19, pass obliquely through a lens *LL*. After passing through the lens the wave fronts are of the shape of *AAA*, Figs. 17 and 18, and the rays pass through a line *C*, Fig. 19, and further on the rays pass through a line *DD*. If the light from the lens *LL*, Fig. 19, is allowed to fall on a screen a short bright line will appear on the screen if the screen passes through *C* or through *DD*. If the screen is between *C* and *DD*, as shown by the line *SS*, Fig. 19, a blurred spot will show on the screen. The light from *O* passing obliquely through *LL* cannot be focused on a screen as a sharp point. The nearest approach to a sharp point is when the screen is midway between *C* and *DD* in which case the blurred spot is a small circle called the *circle of least confusion*.

CHAPTER III.

REFLECTION AND REFRACTION.

29. Regular and diffuse reflection. Refraction. — When light falls upon the surface of a body, it is in part reflected. When reflected light enters the eye from a polished surface, we see the objects from which the light has originally come. On the other hand, we see only the reflecting body if its surface is rough. Reflection from polished surfaces is called *regular reflection*. This kind of reflection is discussed in the following articles. Reflection from rough surfaces is called *diffuse reflection*.

When light enters one medium from another across a polished surface, as for example when light enters glass or water from air, the direction of progression of the light is, in general, altered. This phenomenon is called *refraction*.

30. Index of refraction. — The velocity of light is different in different media. The ratio : $\frac{\text{velocity of light in air}}{\text{velocity of light in given substance}}$ is called the *index of refraction* * of the substance, and is usually represented by the letter μ. The index of refraction of a substance may be determined by the direct measurement of the velocity of light, as, for example, by the method of Foucault. The most convenient method for determining the index of refraction of a substance is described in Article 46.

The velocity of light in a given substance is different for light of different wave-lengths, therefore the index of refraction of a substance varies with the wave-length of the light. The phenomena which depend upon the variation of light velocity with wave-length are described in Chapter VII.

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* The ratio : $\frac{\text{velocity of light in a vacuum}}{\text{velocity of light in a substance}}$ is called the absolute index of refraction of the substance.

31. The application of Huygens' principle to reflection and refraction.—When a light wave approaches the polished surface of a transparent substance, such as glass, a portion of the wave enters the glass and is refracted and a portion of the wave is turned back or reflected by the surface of the glass. The determination of the location of the reflected and refracted waves at any given instant depends upon the application of Huygens' principle (Art. 24) slightly modified as follows: *Each point of the surface of the glass is a center of disturbance at the instant that the advancing wave reaches that point, and a wavelet emanates from this point into the glass at a definite velocity and into the air at a velocity μ times as great*, μ being the index of refraction of the glass.

A wave emanates from a point *O*, Fig. 20, and approaches a reflecting and refracting surface *AB*. At a *given instant* the wave would have reached the position *WW* if it had not encountered the glass. It is required to find at the given instant the position of the reflected wave and the position of the refracted wave. The wavelet from each point *p* of the surface *AB* is partly in glass and partly in air. The part that is in the air is a portion of a sphere of radius *r* and the part that is in the glass is a portion of a sphere of radius r/μ, μ being the index of refraction of the glass, and *r* being the shortest distance from *p* to *WW*, that is, *r* is the radius of the sphere which described about *p* as a center is tangent to *WW*.

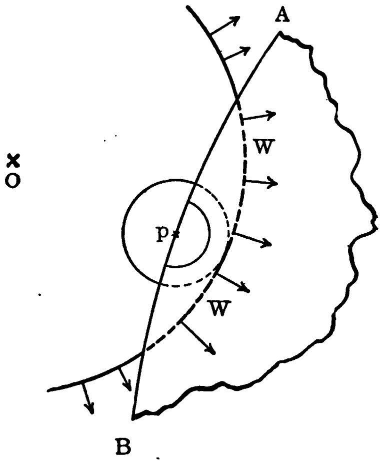

Fig. 20.

The two parts of the wavelet from *p* are shown by the small full-line semicircles in Fig. 20. The dotted semicircle is a continuation of the larger full-line semicircle and it shows that if the wavelet were wholly in air it would be tangent to *WW*.

Fig. 21 shows Huygens' construction for the reflected wave. From each point of the surface of the glass a sphere is described of such radius as to touch *WW*. The envelope of these wavelet spheres is the reflected wave.

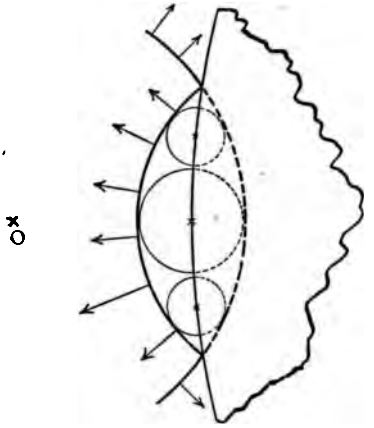

Fig. 21.

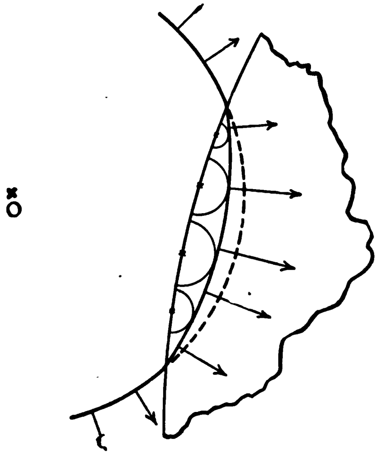

Fig. 22.

Fig. 22 shows Huygens' construction for the refracted wave. From each point of the surface of the glass a sphere is described of which the radius is $1/\mu$ as great as it would have to be to touch *WW*. The envelope of these wavelet spheres is the refracted wave.

32. Reflection of a plane wave from a plane surface. — Consider a plane wave *CD*, Fig. 23, approaching a plane polished surface *AB*. At a *given instant* this wave would have reached the position *WW* if it had not encountered the reflecting surface. It is required to find the position of the reflected wave at the given instant. From each point of the surface *AB* describe a sphere tangent to *WW*. The envelope *W′W′* of these wavelet spheres is the reflected wave. The angle *i* between the incident wave-front and the reflecting surface (or between the incident ray and the normal to the reflecting surface) is called the *angle of incidence.* The angle *r* between the reflected wave front and the reflecting surface (or between the reflected ray and the normal to the reflecting surface) is called the *angle of reflection.* These two angles are evidently equal to each other by the construction of the figure.

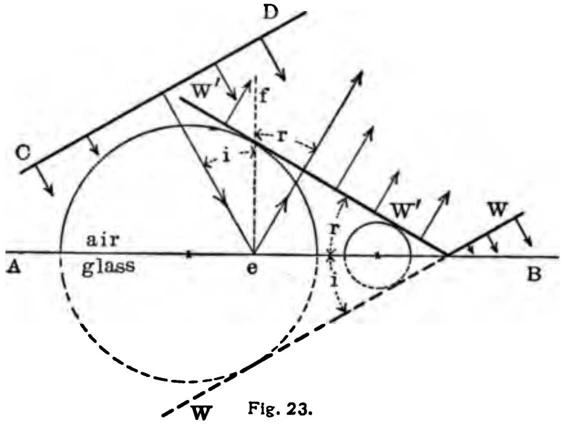

Fig. 23.

33. Reflection of a spherical wave from a plane surface. — Let *O,* Fig. 24, be a luminous point from which spherical waves emanate, let *AB* be a plane reflecting surface, and let *WW* be the position which would be reached at a *given instant* by a spherical

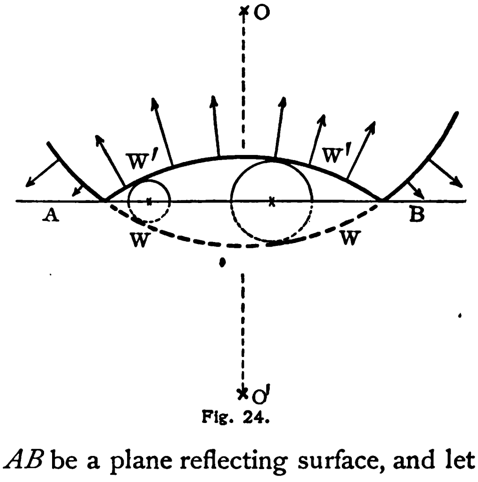

Fig. 24.

wave from *O* were it not for the reflecting surface *AB*. It is required to find the position of the reflected wave at the given instant. From each point of the surface *AB* describe a sphere tangent to *WW*. The envelope *W′W′* of these wavelet spheres is the reflected wave. It is evident from the symmetry of the figure that the reflected wave *W′W′* is a sphere exactly like *WW*. The center of curvature of the reflected wave is *O′*. Therefore, when light from a luminous point *O* is reflected from a plane surface the reflected light appears to come from another fictitious luminous point *O′*. This point *O′* is called the *image* of *O*. The line *OO′* is perpendicular to the plane of the reflecting surface and bisected thereby.

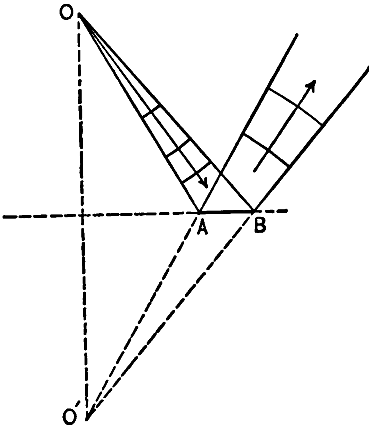

Fig. 25.

Fig. 25 shows the reflection of light from a small plane mirror. The relative positions of *O* and *O′* are the same as if the mirror were large.

34. Image of an object in a plane mirror. — An ordinary object is merely a group of luminous points in so far as its light-giving properties are concerned. Consider such an object *O*, Fig. 26. The light from the various points of this object, after reflection from a plane surface *AB*, or from a portion of such surface, appears to come from a similar group of points *O′*. This group of fictitious luminous points *O′* is called the *image* of the object *O*.

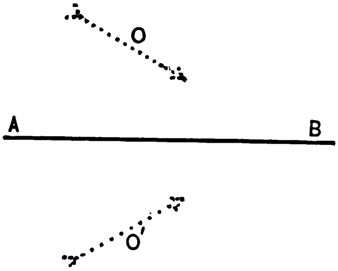

Fig. 26.

35. Reflection from curved surfaces. — Two simple cases of reflection from curved surfaces are the reflection from the paraboloid of revolution and from the ellipsoid of revolution. The reflection from the paraboloid of revolution is shown in Fig. 27. A beam of parallel rays falling upon a paraboloid is concentrated at the focus of the paraboloid, or light emanating from the focus of the paraboloid is reflected by the paraboloid as a beam of parallel rays. The parabolic reflector is much used for giving a beam of parallel rays of light from a light source placed at the focus of the reflector, as in locomotive headlights and in searchlights. The reflection from the ellipsoid of revolution is shown in Fig. 28. Light emanating from one focal point of the ellipsoid is concentrated at the other focal point. Vaulted ceilings sometimes approximate so closely to the ellipsoidal form that a faint sound emanating from one point is concentrated by reflection at another point.

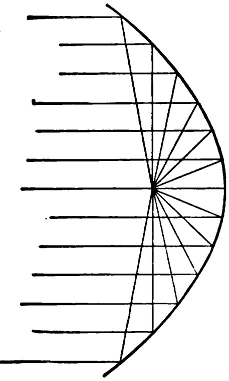

Fig. 27.

The reflection of plane and spherical waves from a spherical surface is in its general aspects an extremely complicated phenomenon. Some of the general features of this kind of reflection will be briefly described in the next article, after which the simplified *approximate* theory of spherical mirrors will be discussed.

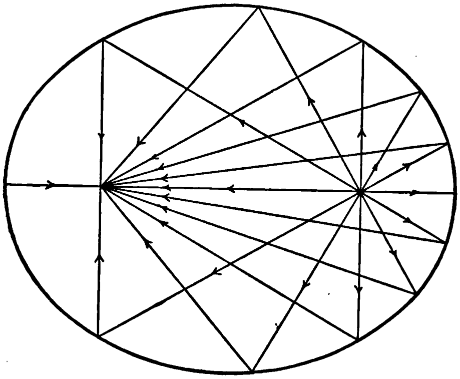

Fig. 28.

36. Reflection of a plane wave from a spherical surface. — Let *MM*, Fig. 29, represent a polished spherical surface with its center of curvature at *R*. A plane wave comes from the right and the line *WW* represents the position this wave would reach at a *given instant* if it were not for the mirror *MM.* It is required to find the position of the reflected wave at the given instant. From each point of the reflecting surface describe a sphere tangent to *WW.* The envelope *W' W'* of these wavelet spheres is the required reflected wave.

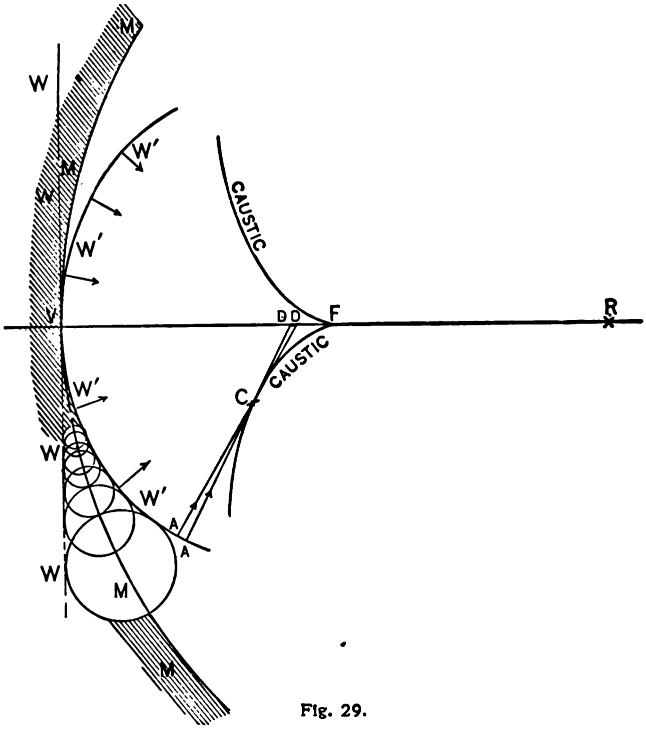

Fig. 29.

In Fig. 30 three successive positions 1, 2, 3, of the reflected wave are shown. The reflected wave is evidently not a spherical wave although a small part of the central portion of the reflected wave approximates very closely to a spherical shape with center of curvature at the point *F*, Fig. 29, which is midway between *V* and *R*. (Art. 38.)

As the reflected wave *W' W'* advances towards the right, as wn in Fig. 30, the two cusps or sharp points on the wave trace a curved line called the *caustic*. This caustic is the locus of the centers of curvature of the various parts of the reflected wave. Thus the small portion *AA*, Fig. 29, of the reflected wave is sensibly a portion of a circle with its center at *C* on the caustic curve. The caustic curve is a region of intense illumination inasmuch as the disturbance represented by each portion of the reflected wave is concentrated at the center of curvature of that portion. The caustic curve is therefore visible as a sharp bright line when the light reflected from a curved surface is thrown obliquely on a flat sheet of paper.

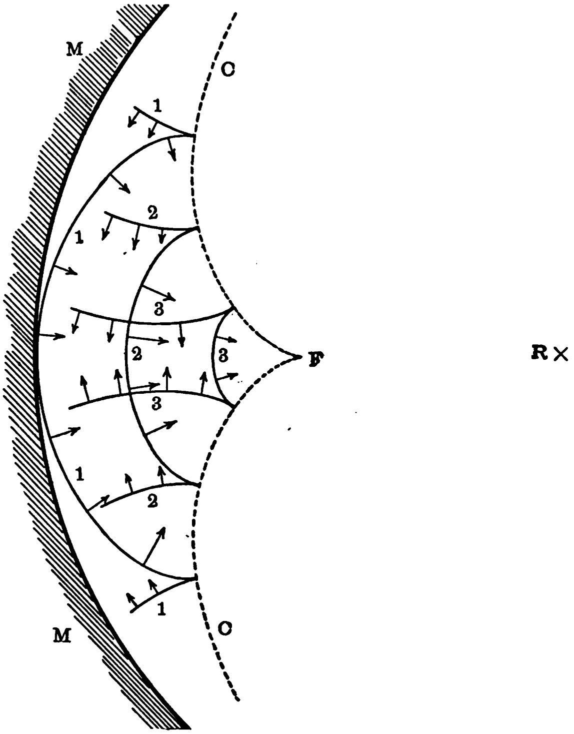

Fig. 30.

The pencil of rays drawn from a small portion *AA*, Fig. 29, of the reflected wave constitutes an astigmatic pencil of which one focal line is at C perpendicular to the plane of the paper and the other focal line is a short portion DD of the line VR.

37. The spherical mirror of small aperture. — A mirror has a small aperture when the diameter of the mirror is small compared with the diameter of the sphere of which the mirror is a portion. Thus the central portion of the mirror MM, Fig. 29, would be a mirror of small aperture.

A spherical mirror with its reflecting surface on its concave side is called a *concave* mirror. When the reflecting surface is convex the mirror is called a *convex* mirror. Convex mirrors are of little practical use and therefore it is not worth while to discuss them in detail.

The *axis* of a mirror is the line drawn from the center of the mirror to the center of curvature of the surface of the mirror. Thus the line VR is the axis of the mirror shown in Fig. 29.

A pencil of parallel rays of light, parallel to the axis of a concave mirror of small aperture, is concentrated by the mirror at a point, F, Fig. 29, called the *principal focus* of the mirror. The distance from the mirror to its principal focus is called the *focal length* of the mirror.

Any pencil of parallel rays not very greatly inclined to the axis of a concave mirror of small aperture is concentrated by the mirror at a point lying in a plane perpendicular to the axis of the mirror and including the principal focus F of the mirror. This plane is called the *principal focal plane* of the mirror.

Conjugate foci. — Two points so situated that light from one is concentrated at, or appears to come from, the other after reflection from a concave mirror are called conjugate foci, or one point is said to be conjugate to the other.

When a point lies in the axis of a mirror its conjugate also lies in the axis. When a point is moved away from the axis in a plane perpendicular to the axis the conjugate to the point moves away from the axis in a plane perpendicular to the axis. Two planes perpendicular to the axis are called *conjugate planes* when the points in one plane have their conjugates in the other plane.

The discussion of the spherical mirror and of lenses depends upon the following geometrical theorem: Consider an arc of a circle of which the chord is d and the versed sine is h; the radius of the circle being R, as shown in Fig. 31. Then

$$R = \frac{d^2}{8h} + \frac{h}{2} \qquad (a)$$

or, when h is small compared with d we have, approximately

$$R = \frac{d^2}{8h} \qquad (3)$$

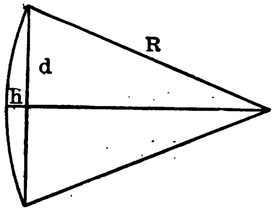

Fig. 31.

The proof of this equation is as follows: Consider the right triangle in Fig. 31, of which the sides are R, $d/2$, and $R-h$. From this triangle we have

$$R^2 = (R-h)^2 + \left(\frac{d}{2}\right)^2$$

from which equation (a) results by simple transformation.

38. Proposition. — *The focal length, p, of a mirror is equal to one half the radius of curvature of the mirror.*

Proof. — Let d be the diameter of the mirror and R its radius of curvature. Consider the instant when an incident plane wave has just reached the edge or rim of the mirror. Let h be the distance that the central portion of the plane wave is from the mirror. Then from equation (3) we have

$$R = \frac{d^2}{8h} \qquad (a)$$

The central portion of the wave has still the distance h to travel before it reaches the mirror, and the edge portions of the wave are just being reflected. Therefore, when the central portion reaches the mirror the edge portions of the wave will have traveled a distance h away from the edge of the mirror so that the reflected wave will be a portion of a sphere (sensibly) of which the chord is approximately equal to d and the versed sine is $2h$ so that from equation (3) the radius of curvature p of the reflected wave is:

$$p = \frac{d^2}{16h} \qquad (b)$$

From equations (a) and (b) we have

$$p = \frac{R}{2} \qquad (4)$$

This radius of curvature, p, of the reflected wave is the distance from the mirror to the point where the wave will be concentrated, that is, p is the principal focal length of the mirror. Equation (4) shows that the focal length of a concave mirror is equal to half the radius of curvature of the mirror.

39. Proposition.—*The relation between the focal length p of a concave mirror and the respective distances a and b of a pair of conjugate points or foci from the mirror is:*

$$\frac{1}{p}=\frac{1}{a}+\frac{1}{b} \qquad (5)$$

*Proof.**—Let d be the diameter of the mirror, R its radius of curvature, and h the versed sine corresponding to chord d and radius R. Then from equation (3) we have

$$R=\frac{d^2}{8h} \quad \text{or} \quad \frac{1}{R}=\frac{8h}{d^2} \qquad (a)$$

Consider a spherical wave which has just reached the edge or rim of the mirror from a luminous point distant a from the mirror, and let k be the versed sine of this spherical wave corresponding to chord d. Then from equation (3) we have

$$a=\frac{d^2}{8k} \quad \text{or} \quad \frac{1}{a}=\frac{8k}{d^2} \qquad (b)$$

The central portion of the above wave still has the distance $h-k$ to travel before it reaches the mirror. Therefore when the central portion of the wave reaches the mirror the edge portions of the wave will have been reflected and will have traveled to a distance $h-k$ away from the edge of the mirror. Therefore the versed sine of the reflected wave corresponding to chord d will be $h+(h-k)$ or $2h-k$ so that from equation (3) the radius of curvature b of the reflected wave or the distance from the mirror to the conjugate to a is:

$$b=\frac{d^2}{16h-8k} \quad \text{or} \quad \frac{1}{b}=\frac{16h-8k}{d^2} \qquad (c)$$

From equations (a), (b) and (c) we have

$$\frac{1}{b}=\frac{2}{R}-\frac{1}{a} \quad \text{or} \quad \frac{2}{R}=\frac{1}{a}+\frac{1}{b}.$$

But $\frac{2}{R}=\frac{1}{p}$ from equation (4) so that $\frac{1}{p}=\frac{1}{a}+\frac{1}{b}$, which was to be proved.

40. Virtual and real foci.—When light from a point is actually concentrated at the conjugate focus, the conjugate is said to be *real;* when after reflection from the mirror the light merely appears to have come from the conjugate focus, the conjugate is said to be *virtual.* Virtual foci are always back of the mirror. Figs. 32 and 33 illustrate real and virtual foci.

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* It is easier and better for the student to picture in his mind the geometrical relations involved in this proof than it is for him to see these relations in a drawing.

41. Proposition. — *A pair of conjugate foci which do not lie on the axis of a concave mirror lie on a straight line passing through the center of curvature of the mirror.*

Proof. — The rays from a luminous point *a*, Fig. 34, all pass through *b*, the conjugate to *a*, after reflection from a mirror. Consider that particular ray from *a* which passes through the center of curvature *R* of the mirror. This ray strikes the surface of the mirror normally and after reflection it again passes through *R* and *a*; but it also passes through *b*, the conjugate to *a*; therefore, *a*, *R* and *b* lie on a straight line.

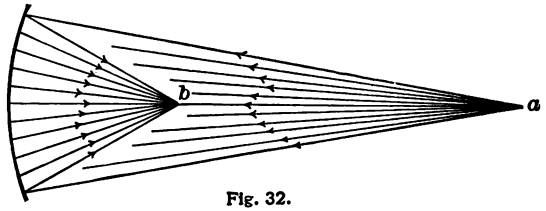

Fig. 32.

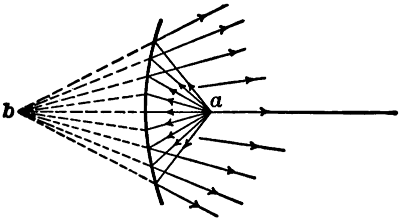

Fig. 33.

42. Geometrical construction for determining the position of the conjugate to a given point. — Let *MM*, Fig. 34, be a concave mirror. Let *R* be its center of curvature, and *F* its principal focus, and let *a* be a given luminous point. Draw from *a* a straight line through *R*. The point *b* which is the conjugate to *a* lies on this line. Draw from *a* a ray parallel to the axis of the mirror. This ray after reflection passes through *F*, since all rays parallel to the axis of a mirror are reflected to the principal focus. It also passes through *b*. Therefore the point *b* is at the intersection of *aR* and *XF*.

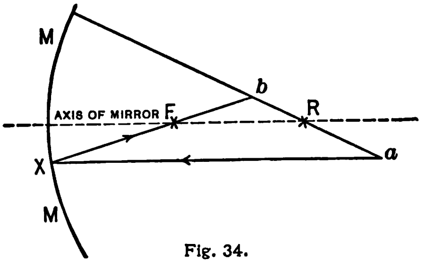

Fig. 34.

43. Formation of images by concave mirrors. *Geometrical construction.* — The group of points which are conjugate to the respective points of an object constitute an *image* of the object. When these conjugate points are real (Art. 40) the image is said to be real. When the conjugate points are virtual (Art. 40) the image is said to be virtual. Virtual images are always behind the mirror, real images are always in front of the mirror.

Fig. 35 shows a concave mirror *MM*, an object *aa′*, and the real image *bb′* of the object. The center of curvature of the mirror is at *R* and its principal focus at *F*.

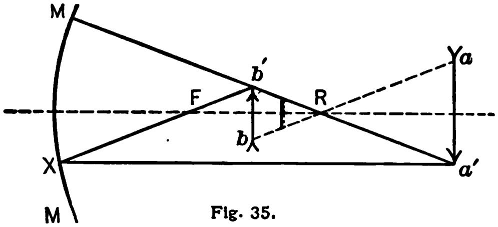

Fig. 35.

From the point *a′*, Fig. 35, those rays only are shown which serve to locate the conjugate *b′* as explained in Art. 42. [In all diagrams of mirrors (and lenses) it is best to draw only those rays which are geometrically significant according to Art. 42. leaving it to the reader's imagination to picture the further details.]

Fig. 36 shows a concave mirror *MM*, an object *aa′*, and the virtual image *bb′* of the object. The construction shown in Fig. 36 is essentially the same as that shown in Fig. 34. Thus from *a* a straight line is drawn through *R*; the conjugate to *a* lies on this line. Also from *a* the ray *aX* is drawn parallel to the axis of the mirror and from *X* the reflected ray *XF* is drawn; the conjugate to *a* lies on *XF*. Therefore, the conjugate to *a* is at the intersection of *Ra* and *XF*. When the object *aa′* is between the mirror and its principal focus *F* then the image *bb′* is virtual and behind the mirror, as shown in Fig. 36. When the object *aa′* is farther from the mirror than the principal focus *F* then the image is real and in front of the mirror as shown in Fig. 35.

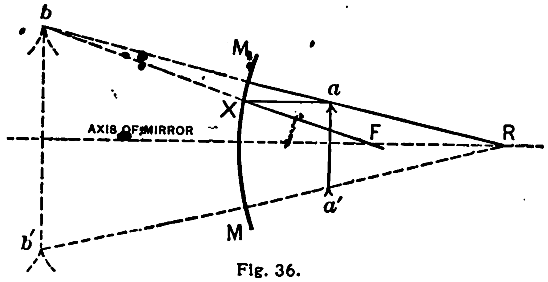

Fig. 36.

44. Linear magnification of image of an object. — The triangles *Raa′* and *Rbb′* in Figs. 35 and 36 are similar. Therefore the length of the object *aa′* is to the length of the image bb' as the distance of object from center of curvature R is to the distance of image from center of curvature R.

45. Refraction of a plane wave at a plane surface. — Consider a plane wave CD, Fig. 37, approaching the plane surface AB of a piece of glass. At a *given instant* this wave would have reached the position WW were it not for the piece of glass. It is required to find the position of the refracted wave at the given instant. From each point of the surface AB describe a sphere of which the radius is d/μ where d is the radius of the sphere which is tangent to WW and μ is the index of refraction of the glass. The envelope of these wavelet spheres (of which only one is shown in the figure) is the required refracted wave.

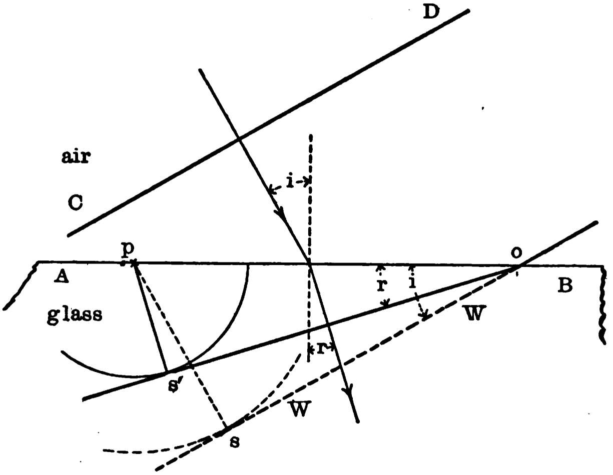

Fig. 37.

The angle i between the incident wave-front and the refracting surface (or between the incident ray and the normal to the refracting surface) is called the *angle of incidence*. The angle r between the refracted wave-front and the refracting surface (or between the refracted ray and the normal to the refracting surface) is called the *angle of refraction*.

From the right triangle pos we have

$$\overline{po} \sin i = d \qquad (a)$$

From the right triangle *pos′* we have

$$\overline{po}\sin r = \frac{d}{\mu} \qquad (b)$$

Dividing equation (*a*) by equation (*b*) member by member we have

$$\frac{\sin i}{\sin r} = \mu \qquad (6)$$

This relation between the index of refraction μ and the angles of incidence and refraction i and r was discovered by Snell and is sometimes called *Snell's law.*

46. Refraction of a plane wave through a prism. — The prism is a portion of a transparent substance such as glass bounded by two plane polished faces inclined to each other. A ray of light passing through a prism is refracted as shown in Fig. 38. The whole angle of deflection a depends upon the refractive index of the material, upon the angle ϕ, and upon the direction of the ray in the prism. When the ray in the prism is equally inclined to the two faces of the prism the deflection a is a minimum.*

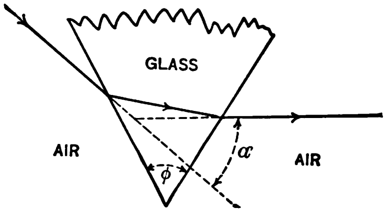

Fig. 38.

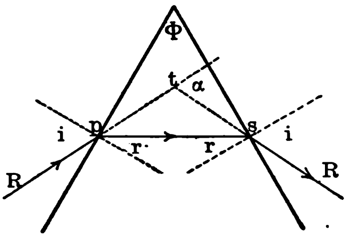

Fig. 39.

Equation for minimum deflection. — The equation for minimum deflection of a ray by a prism is important inasmuch as it is by the measurement of this minimum deflection, the angle ϕ being known, that the index of refraction of a sample of glass is usually determined (Art. 99). A prism is adjusted to give minimum deflection of a ray *RR* as shown in Fig. 39, and the angle a is measured. Then

$$\mu = \frac{\sin \frac{1}{2}(\phi + a)}{\sin \frac{1}{2}\phi} \qquad (6)\,b$$

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* See Preston's "Light," p. 123.

This equation is derived as follows : From equation (6) we have

$$\mu = \frac{\sin i}{\sin r} \qquad (a)$$

From Fig. 39 we have $\phi = 2r$ or

$$r = \tfrac{1}{2}\phi \qquad (b)$$

From the triangle *pst*, Fig. 39, we have $a = 2(i - r)$ or, using equation (*b*), $a = 2i - \phi$; whence

$$i = \tfrac{1}{2}(\phi + a) \qquad (c)$$

Substituting the values of r and i from (*b*) and (*c*) in (*a*) we have equation (6)*b*.

47. Total reflection. — Consider a ray of light passing from glass into air as shown in Fig. 40. In this case, by interchanging i and r equation (6) gives:

$$\sin r = \mu \sin i \qquad (a)$$

in which μ is the index of refraction of the glass and is equal, say, to 1.5.

Now obviously the angle i may have any value whatever and its sine may have any value from zero to unity, but no angle can have a sine greater than unity and the question rises: What does equation (*a*) mean when sin i is greater than $1/\mu$? for in this case the sine of r would be greater than unity according to this equation. As a matter of fact when sin i is greater than $1/\mu$ the incident ray R, Fig. 40, is *totally reflected* and turned back into the glass, and there is no such angle as r since there is no ray in the air. This phenomenon of total reflection occurs when light approaches the bounding surface of any transparent substance from the interior of the substance at an angle of incidence for which $\sin i > 1/\mu$, where μ is the index of refraction of the substance.

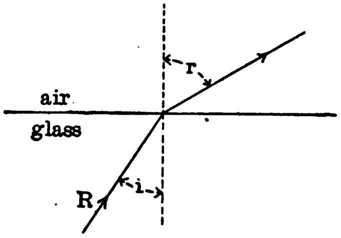

Fig. 40.

Total reflection is the cause of the brilliant silvery appearance of the surface of water in a tumbler when viewed obliquely from below. The well-known lecture experiment of the *illuminated water jet* is based upon the phenomenon of total reflection. In this experiment water escapes from an orifice *o* in a tank as shown in Fig. 41. In the opposite wall of the tank is a lens through which an intense beam of light enters the tank and is concentrated in the orifice *o*. The rays of light strike the boundary of the jet at very oblique incidence and are totally reflected. Passing on further the rays again strike the boundary of the jet at very oblique incidence and are again totally reflected and so on. Fine particles of dust in the jet cause a slight diffusion of the light. A portion of the light so diffused strikes the boundary of the jet at angles of incidence for which total reflection does not occur, and the whole jet appears softly luminous. Where the jet strikes an obstacle, or where it breaks up into drops, the bounding-surface is sufficiently changed in direction to allow the light to escape, thus producing brilliant illumination.

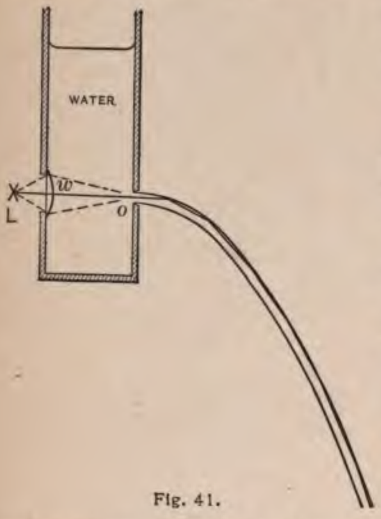

Fig. 41.

The field of a microscope may be illuminated by conveying light along a solid glass rod, one end of which is beneath microscope slide, while the other end is polished flat and is near the lamp flame. The light which enters the rod from the lamp is kept in the rod by total reflection until it reaches the extreme end where it emerges into the air.

48. Refraction of a spherical wave at a plane surface. — The refraction of a spherical wave at a plane surface is rather complicated from the analytical point of view inasmuch as the refracted wave is not spherical. The following discussion is therefore mainly descriptive and most of the statements refer to the details of Figs. 42 and 43, which are made from accurate drawings.

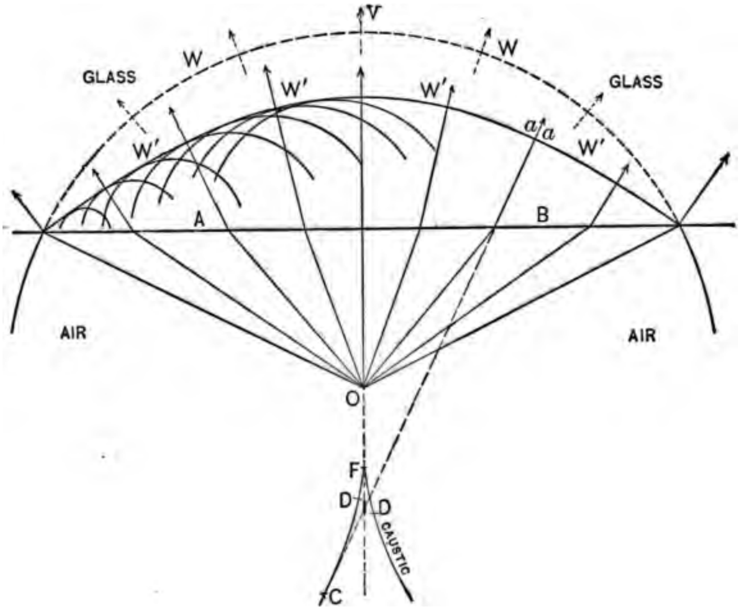
V
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Fig. 42.

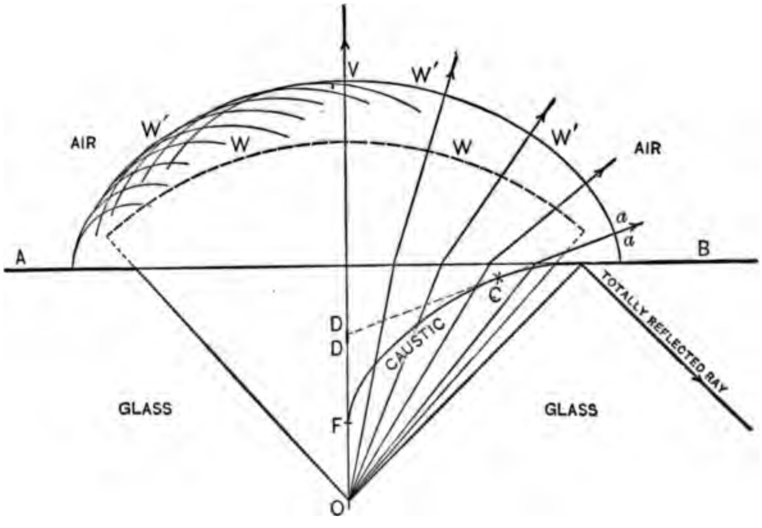
V
W'
AIR
W'
W
W
W'
AIR
a
a
A
B
C
TOTALLY REFLECTED RAY
D
D
CAUSTIC
GLASS
GLASS
F
O

Fig. 43.

A luminous point O, Fig. 42, sends out spherical waves which approach the plane surface AB of a piece of glass. The dotted line WW represents the position which would be reached by one of these spherical waves at a *given instant* were it not for the glass. It is required to find the position of the refracted wave at the given instant. From each point of the surface AB describe a sphere of radius d/μ, where d is the distance from the point on AB to the dotted line WW and μ is the index of refraction of the glass. The envelope $W'W'$ of these wavelet spheres is the required refracted wave. The caustic curve as shown in the figure is the locus of the center of curvature of the various portions of the refracted wave $W'W'$. The rays from a small portion aa of the refracted wave constitute an astigmatic pencil of which one focal line is at C on the caustic curve (perpendicular to the plane of the paper) and the other focal line is a portion DD of the vertical line OV.

A luminous point O, Fig. 43, in (glass or water) sends out spherical waves which pass into air across the plane surface AB. The dotted line WW represents the position which would be reached by one of these spherical waves at a *given instant* were it not for the passage into air. It is required to find the position of the refracted wave at the given instant. From each point of the surface AB describe a sphere of radius μd, where d is the distance from the point on AB to the dotted line WW and μ is the index of refraction of the glass. The envelope $W'W'$ of these wavelet spheres is the required refracted wave. If O, Fig. 43, is a fine luminous point, such as a small globule of mercury at the bottom of a basin of water, it will appear to be at F if viewed from V. If the observer's eyes are at aa the point O will appear to be at C on the caustic curve if the line joining the observer's eyes is vertical; and at D if the line joining the observer's eyes is horizontal.

49a. Refraction at a spherical surface. — The refraction of plane and spherical waves at a spherical surface is a very complicated phenomenon inasmuch as the waves are in general neither spherical nor plane after refraction. Following is a discussion of the one case in which a spherical wave remains spherical after refraction at a spherical surface.

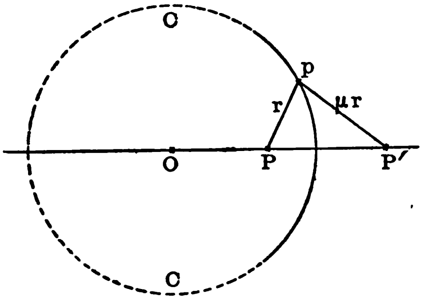

Fig. 44.

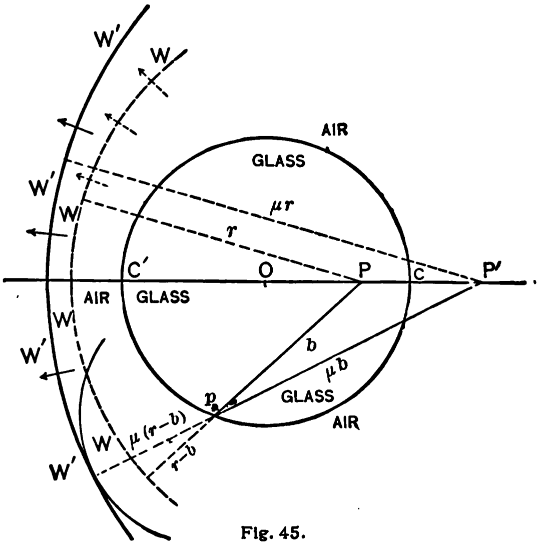

Fig. 45.

Preliminary statement and proposition. — Given two fixed reference points P and P', Fig. 44. Imagine the point p to move so that its distance μr from P' is always μ times as great as its distance r from P. It can be shown that in this case the point p describes the surface of a sphere CC. Let this sphere be made of glass of which the index of refraction is μ and let P be a luminous point inside of the glass sphere. Then the spherical waves which emanate from P and pass to the left through the dotted

43. Formation of images by concave mirrors. *Geometrical construction.* — The group of points which are conjugate to the respective points of an object constitute an *image* of the object. When these conjugate points are real (Art. 40) the image is said to be real. When the conjugate points are virtual (Art. 40) the image is said to be virtual. Virtual images are always behind the mirror, real images are always in front of the mirror.

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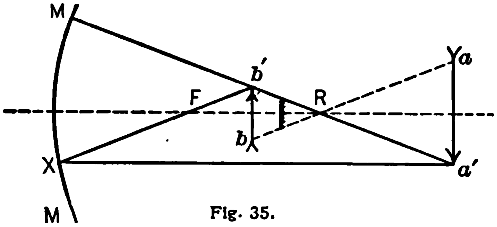

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From the point *a′*, Fig. 35, those rays only are shown which serve to locate the conjugate *b′* as explained in Art. 42. [In all diagrams of mirrors (and lenses) it is best to draw only those rays which are geometrically significant according to Art. 42. leaving it to the reader's imagination to picture the further details.]

Fig. 36 shows a concave mirror *MM*, an object *aa′*, and the virtual image *bb′* of the object. The construction shown in Fig. 36 is essentially the same as that shown in Fig. 34. Thus from *a* a straight line is drawn through *R*; the conjugate to *a* lies on this line. Also from *a* the ray *aX* is drawn parallel to the axis of the mirror and from *X* the reflected ray *XF* is drawn; the conjugate to *a* lies on *XF*. Therefore, the conjugate to *a* is at the intersection of *Ra* and *XF*. When the object *aa′* is between the mirror and its principal focus *F* then the image *bb′* is virtual and behind the mirror, as shown in Fig. 36. When the object *aa′* is farther from the mirror than the principal focus *F* then the image is real and in front of the mirror as shown in Fig. 35.

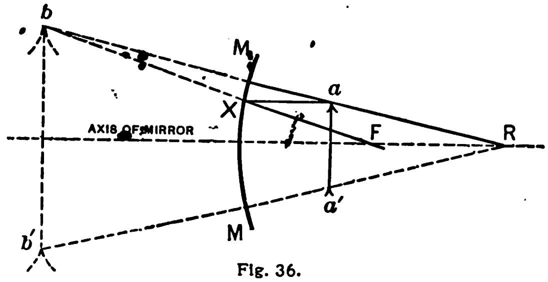

Fig. 36.

44. Linear magnification of image of an object. — The triangles *Raa′* and *Rbb′* in Figs. 35 and 36 are similar. Therefore the length of the object *aa′* is to the length of the image bb' as the distance of object from center of curvature R is to the distance of image from center of curvature R.

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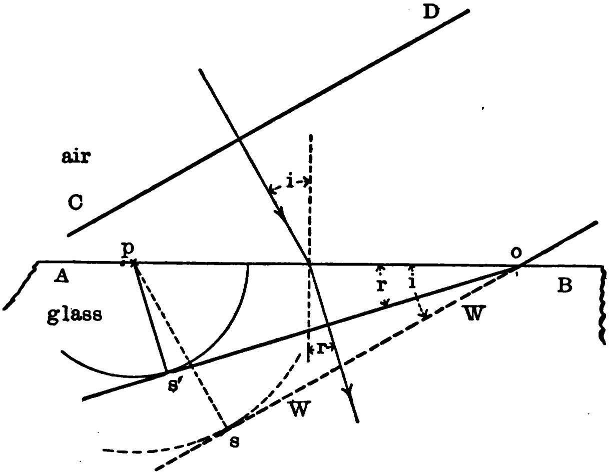

Fig. 37.

The angle i between the incident wave-front and the refracting surface (or between the incident ray and the normal to the refracting surface) is called the *angle of incidence*. The angle r between the refracted wave-front and the refracting surface (or between the refracted ray and the normal to the refracting surface) is called the *angle of refraction*.

From the right triangle *pos* we have

$$\overline{po}\sin i = d \qquad (a)$$

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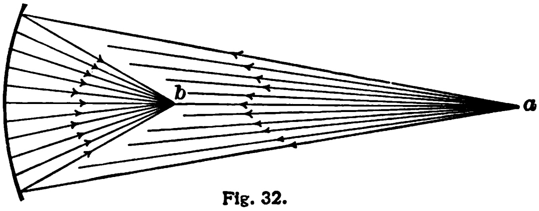
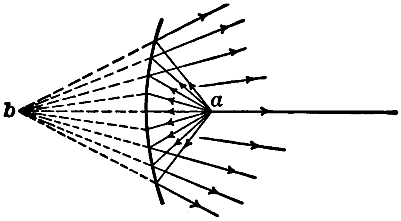


Fig. 32.



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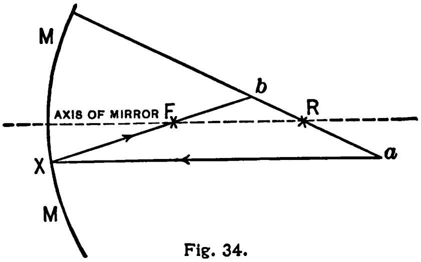
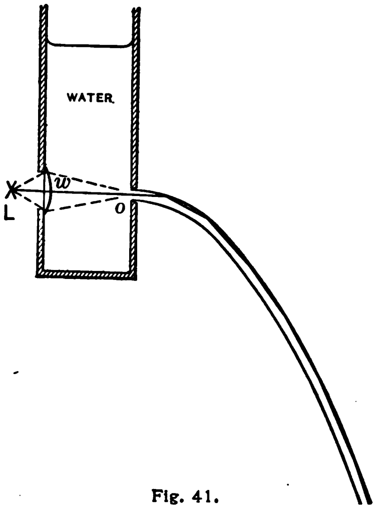


Fig. 34.

this experiment water escapes from an orifice *o* in a tank as shown in Fig. 41. In the opposite wall of the tank is a lens through which an intense beam of light enters the tank and is concentrated in the orifice *o*. The rays of light strike the boundary of the jet at very oblique incidence and are totally reflected. Passing on further the rays again strike the boundary of the jet at very oblique incidence and are again totally reflected and so on. Fine particles of dust in the jet cause a slight diffusion of the light. A portion of the light so diffused strikes the boundary of the jet at angles of incidence for which total reflection does not occur, and the whole jet appears softly luminous. Where the jet strikes an obstacle, or where it breaks up into drops, the bounding-surface is sufficiently changed in direction to allow the light to escape, thus producing brilliant illumination.



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48. Refraction of a spherical wave at a plane surface. — The refraction of a spherical wave at a plane surface is rather complicated from the analytical point of view inasmuch as the refracted wave is not spherical. The following discussion is therefore mainly descriptive and most of the statements refer to the details of Figs. 42 and 43, which are made from accurate drawings.

length of the image bb' as the distance of object from center of curvature R is to the distance of image from center of curvature R.

45. Refraction of a plane wave at a plane surface. — Consider a plane wave *CD*, Fig. 37, approaching the plane surface *AB* of a piece of glass. At a *given instant* this wave would have reached the position *WW* were it not for the piece of glass. It is required to find the position of the refracted wave at the given instant. From each point of the surface *AB* describe a sphere of which the radius is d/μ where d is the radius of the sphere which is tangent to *WW* and μ is the index of refraction of the glass. The envelope of these wavelet spheres (of which only one is shown in the figure) is the required refracted wave.

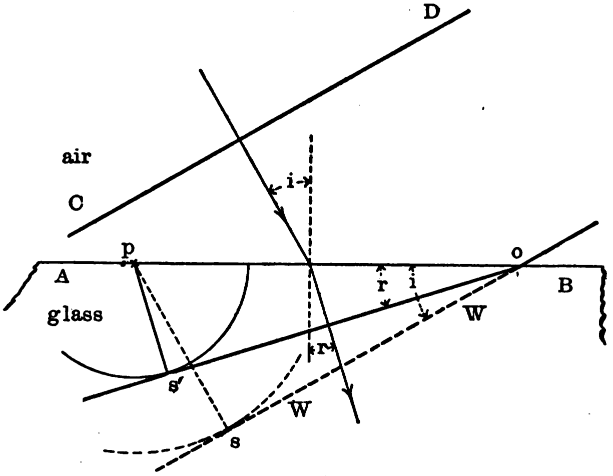

Fig. 37.

The angle i between the incident wave-front and the refracting surface (or between the incident ray and the normal to the refracting surface) is called the *angle of incidence*. The angle r between the refracted wave-front and the refracting surface (or between the refracted ray and the normal to the refracting surface) is called the *angle of refraction.*

From the right triangle *pos* we have

$$\overline{po}\sin i = d \qquad (a)$$

From the right triangle *pos'* we have

$$\overline{po}\sin r=\frac{d}{\mu} \qquad (b)$$

Dividing equation (*a*) by equation (*b*) member by member we have

$$\frac{\sin i}{\sin r}=\mu \qquad (6)$$

This relation between the index of refraction μ and the angles of incidence and refraction i and r was discovered by Snell and is sometimes called *Snell's law.*

46. Refraction of a plane wave through a prism. — The prism is a portion of a transparent substance such as glass bounded by two plane polished faces inclined to each other. A ray of light passing through a prism is refracted as shown in Fig. 38. The whole angle of deflection a depends upon the refractive index of the material, upon the angle ϕ, and upon the direction of the ray in the prism. When the ray in the prism is equally inclined to the two faces of the prism the deflection a is a minimum.*

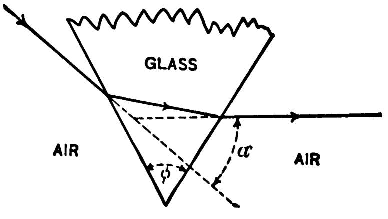

Fig. 38.

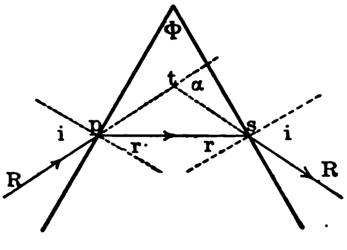

Fig. 39.

Equation for minimum deflection. — The equation for minimum deflection of a ray by a prism is important inasmuch as it is by the measurement of this minimum deflection, the angle ϕ being known, that the index of refraction of a sample of glass is usually determined (Art. 99). A prism is adjusted to give minimum deflection of a ray *R'R'* as shown in Fig. 39, and the angle a is measured. Then

$$\mu=\frac{\sin \frac{1}{2}(\phi+a)}{\sin \frac{1}{2}\phi} \qquad (6)\ b$$

---

* See Preston's "Light," p. 123.

This equation is derived as follows : From equation (6) we have

$$\mu = \frac{\sin i}{\sin r} \qquad (a)$$

From Fig. 39 we have $\phi = 2r$ or

$$r = \tfrac{1}{2}\phi \qquad (b)$$

From the triangle *pst*, Fig. 39, we have $a = 2(i - r)$ or, using equation (*b*), $a = 2i - \phi$; whence

$$i = \tfrac{1}{2}(\phi + a) \qquad (c)$$

Substituting the values of r and i from (*b*) and (*c*) in (*a*) we have equation (6)*b*.

47. Total reflection. — Consider a ray of light passing from glass into air as shown in Fig. 40. In this case, by interchanging i and r equation (6) gives:

$$\sin r = \mu \sin i \qquad (a)$$

in which μ is the index of refraction of the glass and is equal, say, to 1.5.

Now obviously the angle i may have any value whatever and its sine may have any value from zero to unity, but no angle can have a sine greater than unity and the question rises: What does equation (*a*) mean when $\sin i$ is greater than $1/\mu$? for in this case the sine of r would be greater than unity according to this equation. As a matter of fact when $\sin i$ is greater than $1/\mu$ the incident ray *R*, Fig. 40, is *totally reflected* and turned back into the glass, and there is no such angle as r since there is no ray in the air. This phenomenon of total reflection occurs when light approaches the bounding surface of any transparent substance from the interior of the substance at an angle of incidence for which $\sin i > 1/\mu$, where μ is the index of refraction of the substance.

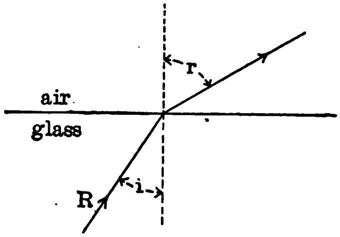

Fig. 40.

Total reflection is the cause of the brilliant silvery appearance of the surface of water in a tumbler when viewed obliquely from below. The well-known lecture experiment of the *illuminated water jet* is based upon the phenomenon of total reflection. In portion of the spherical surface remain spherical after passing into the air and the center of the new spherical waves is the point P'. This fact was discovered by Amici and it is utilized as one of the most important features of the compound microscope.

Proof. — Let the dotted circle WW, Fig. 45, of radius r be the position which the wave from P would reach at a *given instant* if the whole region were glass. Describe a circle $W'W'$ of radius μr with its center at P'. Consider the point p, distant b from P and μb from P'. At the given instant the wavelet from this point has had time to travel a distance $(r - b)$ in glass or $\mu(r - b)$ in air, so that its radius is $\mu(r - b)$. Therefore, since $\mu(r - b) + \mu b = \mu r$, this wavelet touches $W'W'$; $W'W'$ is thus the envelope of all such wavelets; and therefore it is the refracted wave.

49b. The refraction of a narrow homocentric pencil of rays at a small portion of a large spherical surface. — The approximate theory of the concave mirror is based upon the reflection of a narrow pencil of rays from a small portion of a large spherical surface; that is to say from a mirror of small aperture, the pencil of rays being only slightly inclined to the axis of the mirror. The approximate theory of lenses, on the other hand, is based upon the refraction of a narrow pencil of rays at a small portion of a large spherical surface. The line AB, Fig. 46, represents a spherical piece of glass with its center at C. The pencil of rays OV which passes nearly along the line OC is called an *axial* pencil. A pencil of rays such as OX which is considerably inclined to the line OC is called an *oblique* pencil.

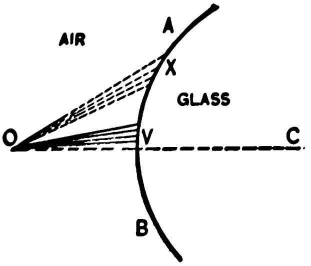

Fig. 46.

A narrow axial homocentric pencil of rays remains sensibly homocentric after refraction at a small portion of a large spherical surface, that is to say light emanating from a point is concentrated at or appears to come from another point after refraction at a spherical surface under the prescribed limitations. A very oblique homocentric pencil of rays such as *OX*, Fig. 46, becomes distinctly astigmatic after refraction at a spherical surface.

CHAPTER IV.

LENSES AND LENS SYSTEMS.

50. The lens is a portion of transparent substance, generally glass, bounded by polished spherical surfaces. The line RR', Figs. 47 and 48, passing through the centers of curvature R and R' of the spherical surfaces of the lens, is called the *axis* of the lens. A lens which is thicker at its center than at its edge is a *converging* lens. A lens which is thinner at its center than at its edge is a *diverging* lens. Fig. 47 shows a converging lens and Fig. 48 shows a diverging lens.

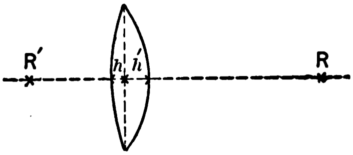

Fig. 47.

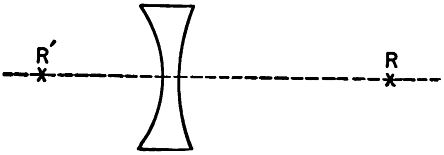

Fig. 48.

The surface of a lens is shaped by grinding the lens and a matrix together with every possible variety of sliding motion. The lens and matrix are thus brought automatically to almost perfect spherical shape. The polishing is accomplished by using finer and finer grinding material, usually emery, and ending with rouge. The matrix in the earlier stages of the grinding is usually of iron. In the later stages the matrix is lined with a layer of stiff pitch.

A plane or spherical wave is, in general, not spherical after passing through a lens. The errors of lenses are discussed in Chapter VI. The immediately following discussion applies to thin lenses of small aperture and to rays not greatly inclined to the axis of the lens. Thick lenses of small aperture are discussed as

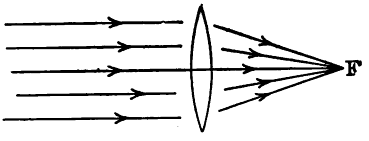

Fig. 49.

51. Principal focus. Principal focal planes. Focal length.—The points (one on either side) at which rays parallel to the axis are concentrated by a converging lens are called the *principal foci* of the lens.

The distance from the center of the lens to a principal focus is called the *focal length* of the lens. A beam of parallel rays inclined to the axis of a converging lens is concentrated at a point in a plane which contains a principal focus and is perpendicular to the axis of the lens. There are two such planes, one on either side of the lens, and they are called the *principal focal planes* of the lens. Thus the plane *AB*, Fig. 50, is a principal focal plane of the lens *LL*. A line drawn from the point *F′*, Fig. 50, to the center of the lens is parallel to the beam of parallel rays which is focussed at *F′*.

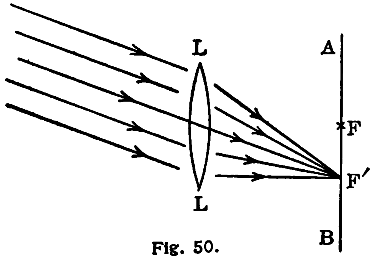

Fig. 50.

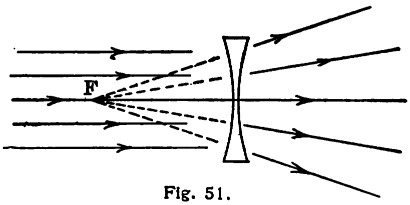

Fig. 51.

The principal focal length of a diverging lens is the distance from the lens to the point *F*, Fig. 51, from which a beam of parallel rays parallel to the axis of the lens appears to emanate after passing through the lens. The action of a diverging lens upon an oblique beam of parallel rays is shown in Fig. 52, in which the line *AB* represents one of the principal focal planes of the lens.

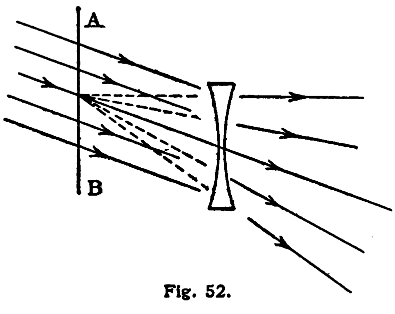

Fig. 52.

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52. Conjugate foci.—Two points so situated that light from one is concentrated at or appears to come from the other after passing through a lens are called *conjugate foci*, or one point is said to be *conjugate to* the other. When a point lies in the axis of a lens its conjugate also lies in the axis. When a point is moved away from the axis in a plane perpendicular to the axis the conjugate to the point moves away from the axis in a plane perpendicular to the axis. Two planes perpendicular to the axis of a lens are called *conjugate planes*, or one plane is said to be conjugate to the other, when the points in one plane have their conjugates in the other plane.

53. Real and virtual foci.—When light from a point is actually concentrated at the conjugate point, the conjugate is said to be *real;* when, after passing through the lens, the light merely appears to have come from the conjugate point, the conjugate is said to be *virtual.* The principal focus of a converging lens is real, the principal focus of a diverging lens is virtual. When the conjugate to a given point is virtual it is on the same side of the lens as the given point.

54. Proposition. — *The focal length of a lens is:*

$$p = \frac{1}{(\mu - 1)\left(\frac{1}{R} + \frac{1}{R'}\right)} \tag{7}$$

in which μ is the refractive index of the glass, and R and R' are the radii of curvature of the surfaces of the lens.*

Proof. — Let d be the diameter of the lens measured to its ideal edge where the surfaces of the lens intersect. Imagine a plane drawn through this thin edge and let h and h' be the distances represented in Fig. 47. Then from equation (3), Art. 37, we have

---

* Either radius R or R' is considered negative when the corresponding surface is concave. The focal length p is considered negative when the lens is diverging.

$$R = \frac{d^2}{8h} \qquad (a)$$

$$R' = \frac{d^2}{8h'} \qquad (b)$$

The thickness of the lens at the center is $h + h'$. *While the central part of a plane wave is traveling the distance* $h + h'$ *through the glass of the lens, the portions of the wave which graze the edge of the lens travel* μ *times as far in air or a distance* $\mu(h + h')$. *Therefore the edge portions of the wave get ahead of the central portion by the amount* $\mu(h + h') - (h + h')$ *and the wave after passing the lens is (sensibly) a portion of a sphere of which the radius of curvature,* p, *is*

$$p = \frac{d^2}{8\left[\mu(h + h') - (h + h')\right]} \qquad (c)$$

according to equation (3), Art. 37. Therefore

$$p = \frac{1}{(\mu - 1)\left[\frac{8h}{d^2} + \frac{8h'}{d^2}\right]}$$

$$= \frac{1}{(\mu - 1)\left(\frac{1}{R} + \frac{1}{R'}\right)}$$

55. Proposition. — Let a and b be the respective distances of a pair of conjugate foci from a lens and p the focal length of the lens. The relation between a, b and p is

$$\frac{1}{p} = \frac{1}{a} + \frac{1}{b} \qquad (8)$$

Proof. — Let d be the diameter of the lens measured to the sharp edge, and $(h + h')$ the thickness of the lens at the center. Consider a spherical wave which has reached the lens from a point distant a from the lens. The portion of this wave which is tr pass through the lens has a chord which is approximately equ to the diameter d of the lens. Let k be the versed sine of this portion of the wave. Then by equation (3), Art. 37, we have

$$a = \frac{d^2}{8k} \tag{a}$$

The central portion of the wave is a distance k ahead of the edge portions before the wave passes through the lens and the action of the lens is to cause the edge portions to gain $(\mu - 1) \times (h + h')$ on the central portion. Therefore after passing through the lens the central portion of the wave is $\{(\mu - 1)(h + h') - k\}$ behind the edge portions. That is the chord of the wave after passing the lens is approximately equal to d and its versed sine is $\{(\mu - 1)(h + h') - k\}$ so that its radius of curvature b is:

$$b = \frac{d^2}{8\{(\mu - 1)(h + h') - k\}} \tag{b}$$

Equation (c) of Art. 54 may be written

$$p = \frac{d^2}{8(\mu - 1)(h + h')} \tag{c}$$

Taking the values of $1/a$ and $1/b$ from equations (a) and (b) and adding them we have

$$\frac{1}{a} + \frac{1}{b} = \frac{8(\mu - 1)(h + h')}{d^2}$$

which by equation (c) is equal to $1/p$.

Remark. — Equation (8) applies to diverging lenses as well as to converging lenses. For diverging lenses p is negative. It is also to be remembered that a or b are positive or negative according as they refer to real or virtual foci.

56. Proposition. — *A pair of conjugate points which do not lie on the axis of a lens lie on a straight line passing through the center of the lens.*

Proof.—The rays from a luminous point a after passing through the lens *all* pass through the point b which is the conjugate to a. Consider that particular ray from a which passes through the center of the lens as shown in Fig. 53. The two surfaces of the lens are parallel to each other where this ray passes through them, therefore the transmitted ray is parallel to the incident ray. The transmitted ray has, however, been displaced sidewise by the lens, but since the lens is assumed to be extremely thin this side displacement is negligible so that incident ray and transmitted ray are sensibly one and the same straight line. Now the point *a* lies on the incident ray and the point *b* lies on the transmitted ray so that *a*, *b*, and the center of the lens are sensibly colinear.

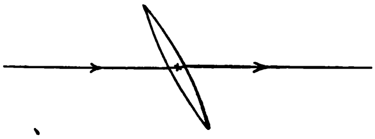

Fig. 53.

57. Geometrical construction for determining the conjugate to a given point.—*LL*, Fig. 54, represents a thin lens, *F* its principal focus, and *a* a given point. To determine the conjugate to *a* draw a straight line from *a* through the center of the lens *C*.

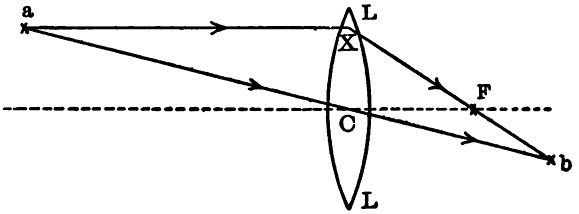

Fig. 54.

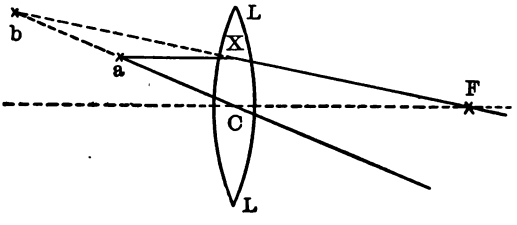

Fig. 55.

Next draw the ray *aX* parallel to the axis of the lens. This ray after passing through the lens must pass through the principa focus *F*. Therefore the line *XF* is the position of the ray *aX* after passing through the lens. All rays from *a* pass through *b* after passing through the lens. Therefore *b* lies on *aC* and on *XF*; that is to say, *b* is at the intersection of *aC* and *XF*.

Fig. 55 shows the construction for the virtual conjugate *b* of a given point *a* in case of a converging lens.

Fig. 56 shows the construction for the conjugate *b* of a given point *a* in case of a diverging lens.

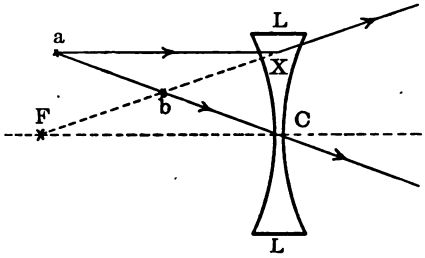

Fig. 56.

Remark. — Figs. 54 to 59 apply to very thin lenses. Therefore the whole deflection of a ray by the lens is represented in these figures as being produced at the central plane of the lens.

58. Formation of images by lenses. — *Geometrical construction.* — The group of points which are conjugate to the respective points of an object constitute an *image* of the object. When these conjugate points are real (Art. 53) the image is said to be real; when these conjugate points are virtual (Art. 53) the image is said to be virtual. Real images are always on the side of the lens opposite to the object, virtual images are always on the same side of the lens as the object.

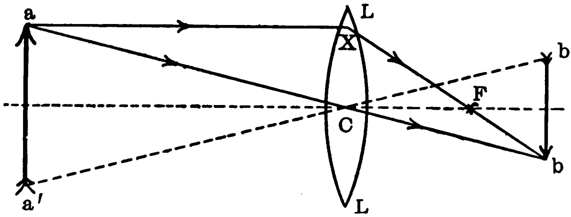

Fig. 57.

Fig. 57 shows a converging lens, an object *aa′*, and the real image *bb′* of the object. The principal focus of the lens is at *F*. From the point *a*, Fig. 57, those rays only are shown which serve to locate the conjugate *b*. [In all diagrams of lenses (and mirrors) it is best to draw only those rays which are geometrically significant leaving it to the reader's imagination to picture the further details.]

Fig. 58 shows a converging lens, an object *aa'*, and the virtual image *bb'* of the object. Whenever an object is between a converging lens and its principal focus the image of the object is virtual.

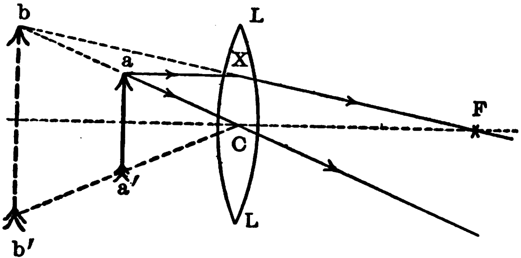

Fig. 58.

Fig. 59 shows a diverging lens, an object *aa'*, and the virtual image *bb'* of the object. A diverging lens always forms a virtual image of an object.

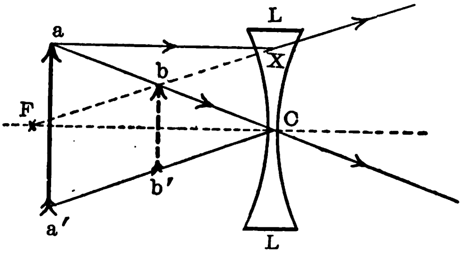

Fig. 59.

59. Linear magnification of image of an object. — The triangles *aCa'* and *bCb'*, Figs. 57, 58, and 59, are similar. Therefore the length *aa'* of the object is to the length *bb'* of the image as the distance of the object from the center of the lens is to the distance of the image from the center of the lens.

60. Lens systems. Compensation of imperfections. — A number of lenses used together constitute a lens system. When the centers of curvature of the various spherical surfaces of the lenses of a system lie on one and the same straight line, the system is called a *centered system.*

Lens systems are used in practice instead of simple lenses in order that the very appreciable imperfections of simple lenses may be obviated. This obviation of the imperfections of simple lenses by combining a number of them into a system is accomplished by so designing the system that the imperfections of one lens are balanced or compensated by opposite imperfections of another lens. The theory of compensated lens systems is outlined in chapter VI.

The following articles have to do with certain fundamental properties of lens systems.

61. Propositions. — (*a*) A narrow pencil of rays from any luminous point O, in or near the axis * of a lens system is sensibly homocentric after passing through the system, and is concentrated at or appears to have come from a point O', called the conjugate to O.

Proof. — According to Art. 49*b*, a narrow pencil near the axis will be homocentric after refraction at the first spherical surface. The resulting homocentric pencil will be homocentric after refraction at the second spherical surface, and so on.

(*b*) A group of luminous points (an object) near the axis in a plane perpendicular to the axis has as its image (with definite magnification, positive or negative) a similar group of points near the axis in another plane perpendicular to the axis. Such planes are called *conjugate planes.*

Proof. — An object has an image formed in accordance with this proposition by the first refracting surface. An image of this image is produced by the second refracting surface, and so on.

Corollary 1. — Any incident ray passing through a point O passes upon emergence through O', the conjugate to O. For by proposition (*a*), *all rays* emanating from or passing through O pass through O' upon emergence. An incident ray and its position upon emergence are called *conjugate rays.*

Corollary 2. — Two incident rays intersecting at O, intersect upon emergence at O'.

Corollary 3. — Consider an incident ray passing through the points O and p. This ray upon emergence passes through O' and p'.

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* In the following figures rays are drawn which make considerable angles with the axis for the sake of clearness.

serve to locate the conjugate *b*. [In all diagrams of lenses (and mirrors) it is best to draw only those rays which are geometrically significant leaving it to the reader's imagination to picture the further details.]

Fig. 58 shows a converging lens, an object *aa'*, and the virtual image *bb'* of the object. Whenever an object is between a converging lens and its principal focus the image of the object is virtual.

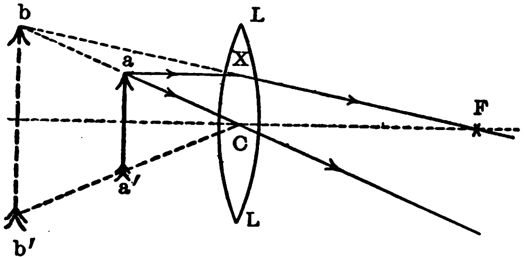

Fig. 58.

Fig. 59 shows a diverging lens, an object *aa'*, and the virtual image *bb'* of the object. A diverging lens always forms a virtual image of an object.

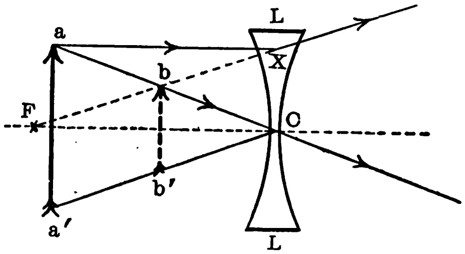

Fig. 59.

59. Linear magnification of image of an object. — The triangles *aCa'* and *bCb'*, Figs. 57, 58, and 59, are similar. Therefore the length *aa'* of the object is to the length *bb'* of the image as the distance of the object from the center of the lens is to the distance of the image from the center of the lens.

Let r, Fig. 63, be an incident ray from the right, and r' its conjugate. Let R', colinear with r, be an incident ray from the *left*, and R its conjugate. The rays r and R intersect at O, and the rays r' and R', conjugate to r and R respectively, intersect at O'. Therefore O and O' are conjugate points, and since they are at the same distance from the axis, it follows that the magnification for the conjugate planes P and P' is unity, and that P and P' are the required principal planes of the system. The distance F to P is called the *right focal length* of the system. The distance F' to P' the *left focal length* of the system. The system represented in Figs. 60, 61, 62, 63, 65 and 66 is a converging system. The two focal lengths of a lens system have a ratio equal to the ratio of the refractive indices of the media, in which the principal foci are situated. In most lens systems used in practice the two focal lengths are equal, since both principal foci are in air.

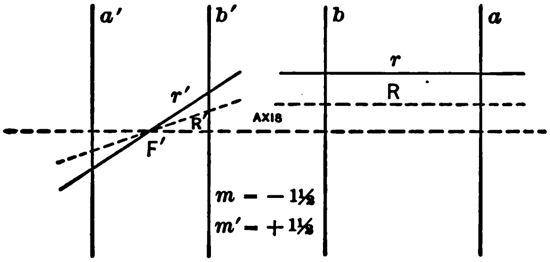

Fig. 62.

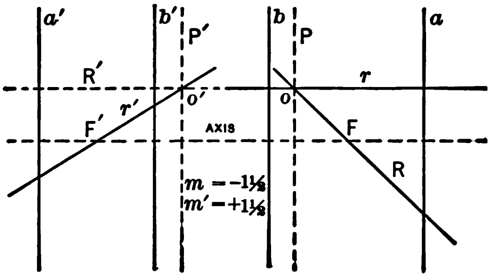

Fig. 63.

65. *Example.*—Fig. 64 shows to scale the actual positions of the principal planes P and P', and of the principal foci F and F' of a symmetrical bi-convex lens — not of infinitesimal thickness—the glass of which has a refractive index of 1.50. The figure also shows the geometrical construction for determining the position of the image of an object. Draw the ray r parallel to the axis from O to the principal plane P', thence after emergence through the focus F'. Draw the ray R from F through O to the principal plane P, thence after emergence parallel to the axis. The conjugate of O is at the intersection of r' and R'. It is marked O'.

Remark 1. — If an emergent ray be reversed, it will retrace its path. If, therefore, the ray r' is conjugate to the ray r, then r is conjugate to r'; and if the point O' is conjugate to O then O is conjugate to O'.

Remark 2. — Any specification which fixes the positions upon emergence of given incident rays is a complete specification of the lens system.

62. Specification of a lens system. — The action of a lens system is completely specified when the positions of two pairs of conjugate planes are given, together with the magnification associated with each pair.

Proof. — Let aa' and bb', Fig. 60, be two given pairs of conjugate planes. Let the magnification $a \mid a'$ be m, and he magnification $b \mid b'$ be m'. Given an incident ray r' (from the left), cutting the planes a' and b' at p' and q', as shown. The transmitted ray r must pass through the points p and q, which are conjugate to p' and q' respectively. The transmitted ray is thus fixed, and the action of the lens system upon the ray r' is completely determined by the given data.

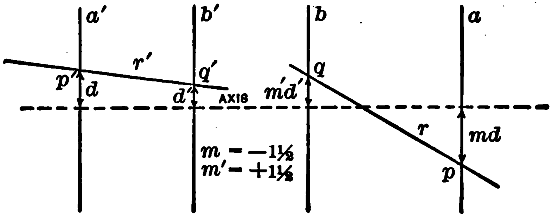

Fig. 60.

63. Principal foci of a lens system. — Consider an incident ray r', Fig. 61, coming from the left, parallel to the axis. The conjugate ray r is as shown. If the distance d changes, the distances md and $m'd$ change in the same ratio, as shown by the dotted rays, R and R', and the point F remains fixed *Therefore all rays from the left, parallel to the axis, pass through F, or seem to have come from F after emergence.* This point F is called the (right) *principal focus* of the system. Figure 62 shows the construction for the left principal focus F'.

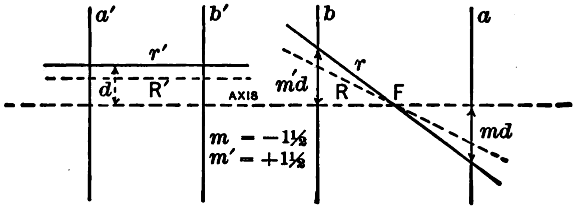

Fig. 61.

64. Principal planes of a lens system. — That pair of conjugate planes for which the magnification is plus one (+ 1) are called the *principal planes* of the system. The following discussion shows that there is always such a pair of planes, and shows their location relative to the given conjugate planes aa' and bb'.

67. The nodal points of a lens system are the two conjugate points in the axis such that any ray passing through them is, upon emergence, parallel to its direction upon incidence. Consider an object in the plane Q, Fig. 65, and its inverted image in the plane Q'. Imagine a ray r (not shown in the figure) passing from the point q to the nodal point N (not shown) and its conjugate* ray r' passing from the nodal point N' to q'. The image, being of same size as the object and inverted, and r being, by definition of nodal points, parallel to r', it must be that N and N' are at equal distances and in *opposite directions* from Q and Q' respectively. Similarly, N and N' are at equal distances and in the *same direction* from P and P' respectively, as shown in Fig. 66. Let x be the distance of N and N' to the left of P and P' respectively, and y the distance of N and N' from Q and Q' respectively. Then, since $Q'P' = 2f'$ and $QP = 2f$,

we have and
$$\left.\begin{aligned} x + y &= 2f' \\ y - x &= 2f \end{aligned}\right\} \qquad \text{(i)}$$

whence
$$\left.\begin{aligned} x &= f' - f \\ y &= f' + f \end{aligned}\right\} \qquad \text{(ii)}$$

It remains to be shown that N and N', the positions of which are determined by (ii), are conjugate points. Draw any parallel lines r and r', Fig. 66, through N and N'. Then, since $NQ = N'Q'$ and $NP = N'P'$, we have $e = e'$ and $d = d'$, so that the points p' and q' are the conjugates to the points p and q; and r and r' are conjugate rays. The same would be true of another pair of parallel lines R and R' (not shown) passing through N and N'. R and r intersect at N, and their conjugate rays R' and r' intersect at N', so that N and N' are conjugate points.

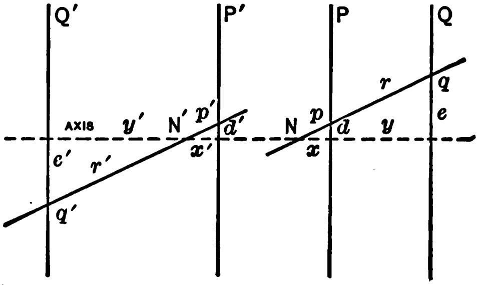

Fig. 66.

Corollary. — Any object and its image subtend equal angles, as seen from the respective nodal points of a lens system.

Remark.—When the right and left focal lengths of a system are equal, the nodal points lie in the principal planes P and P'.

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* It being assumed that N and N' are conjugate points.

CHAPTER V.

SIMPLE OPTICAL INSTRUMENTS.

68. The eye. — This organ is shown in its essential features in Fig. 67. The tough membrane of the eyeball is sharply curved and transparent in front, forming the *cornea C*. Between the cornea and the crystalline lens *L* is a clear, watery fluid, the *aqueous humor*. Behind the crystalline lens, and filling the remainder of the eyeball, is a clear semi-fluid substance, the *vitreous humor*. The front surface of the cornea and the surfaces separating the aqueous humor, crystalline lens, and vitreous humor are sensibly spherical and constitute a lens-system which projects an image of external objects upon a nervous membrane, the *retina*, at the back of the eye. The retina consists of a vast number of minute end organs of nerve fibers which come together, forming the *optic nerve O*, and lead to the brain.

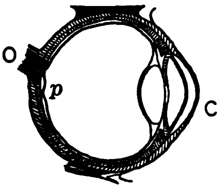

Fig. 67.

Two luminous points can be seen as two so long as their images on the retina do not fall upon one and the same end organ. Over a large portion of the retina these end organs are relatively sparse, but in the thin portion *p*, called from its color the *yellow spot*, they are very closely packed together. An object can be seen sharply only when its image falls upon this spot. The line passing through the center of the eye lenses and the center of the yellow spot is called the *axis* of the eye.

Accommodation. — For distinct vision, the image on the retina must be sharply defined. The focal length of the eye lenses, to give a sharply defined image of an adjacent object upon the retina, must be different from their focal length to give a sha

The magnifying power of a microscope is defined as the quotient of the apparent size (or visual angle) of an object as seen with the microscope, divided by the apparent size (or visual angle) of the object as seen with the naked eye at a distance of 25 centimeters.

73. Proposition.—The magnifying power of a magnifying glass is:

$$m = \frac{25}{p} + 1 \tag{9}$$

in which p is the focal length of the magnifying glass in centimeters. The eye is supposed to be accommodated for a distance of 25 centimeters.

Proof.—Let O, Fig. 70, be the object and i its virtual image formed by the magnifying glass. The eye being accommodated for 25 centimeters, the object will be moved until the image i is 25 cm. from the eye. The distance from the lens to the eye may be neglected, so that the distance b of the image from the lens is 25 cm. Since the distance from a lens to a virtual image is considered as negative, we have, from equation (8), Art. 55:

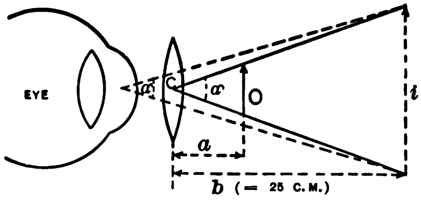

Fig. 70.

$$\frac{1}{p} = \frac{1}{a} - \frac{1}{25} \tag{a}$$

whence

$$a = \frac{25p}{25 + p} \tag{b}$$

Now, the visual angle of the image i is the same as the angle at C subtended by the object O since the distance from the magnifying glass to the eye is neglected. *This angle, a, is as many times as great as the angle that O would subtend at a distance of 25 centimeters, as 25 is greater than a, Fig. 70;* that is

$$m = \frac{25}{a} \tag{c}$$

Substituting the value of a from equation (b) in equation (c) we have equation (9).

Remark 1. — When the eye is accommodated for parallel rays b, Fig. 70, is equal to infinity, a is equal to p, and m is equal to $25/p$.

Remark 2. — A high-power magnifying glass must have very short focal length, and it must be held very near to the object to be examined. Furthermore a short focal length lens is necessarily small in diameter and the field of view in such a magnifying glass is extremely limited. Therefore high-power magnifying glasses are inconvenient and unsatisfactory. For magnifying powers exceeding 20 or 30 diameters the *compound microscope* is to be preferred.

74. The compound microscope consists of a lens A, Fig. 71, which forms an enlarged real image i of an object, and a magnifying glass B for viewing this image. The lens A is called the *object glass* or *objective* and the lens B is called the *eye-piece.* In modern high-grade microscopes the objectives and eye-pieces consist of compensated lens systems as explained in Chapter VI.

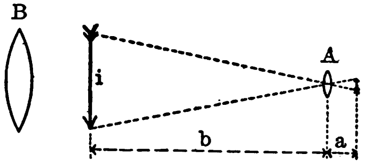

Fig. 71.

Remark. — The dotted lines in Fig. 71 are not intended to represent rays but only to show geometrical relations.

75. Proposition. — The magnifying power of a compound microscope is:

$$m = \frac{b}{a}\left(\frac{25}{p} + 1\right) \qquad (10)$$

in which a and b are the respective distances of object and image from the center of the object glass, as shown in Fig. 71, and p is the focal length of the eye-piece.

Proof. — The image is b/a times as large as the object, and the eye-piece, being a magnifying glass, makes this image appear image of a distant object. This necessary variation of focal length is provided for by the action of muscles, attached to the edge of the crystalline lens, the contraction of which causes the lens to become thinner at the center. This action is called *accommodation.* Ordinarily the eye has power of accommodation for objects at any distance greater than about 15 centimeters from the eye. The distance of most distinct vision, for most individuals, is about 25 centimeters.

The imperfections of the eye. — Some persons can accommodate to distant objects only with great effort, or not at all; such persons are said to be *nearsighted.* Persons who have like difficulty in accommodating to near objects are said to be *farsighted.* Nearsightedness is relieved by the use of spectacles with diverging lenses, farsightedness by the use of spectacles with converging lenses. Nearsightedness (or farsightedness) may be due to an abnormal elongation (or shortening) of the eyeball in the direction of the axis; or to an abnormally short focal length (or long focal length) of the eye lenses. Inaccurate centering of the eye lenses, and more or less deviation from true spherical shape of the various refracting surfaces, produce *astigmatism.* This is sometimes so pronounced as to hinder sharp vision. Such imperfection is corrected by the use of spectacles having cylindrical surfaces.

69. Apparent size of objects. Visual angle. — An object appears large when its image covers a large portion of the retina. Lines drawn from the extremities of the object through the center* of the eye lenses, as shown in Fig. 68, pass through the extremities of the image. The angle a, between these lines is

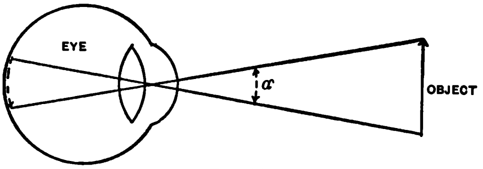

Fig. 68.

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* Strictly, those lines pass through the extremities of the image, which are drawn through the *posterior nodal point* of the lens-system of the eye, parallel respectively to lines drawn from the extremities of the object to the *anterior nodal point.* See Art. 67.

called the *visual angle* of the object, and *it is taken as the measure of the apparent size of the object.* It is important to have in mind a definite numerical measure of apparent size when we come to the discussion of the magnifying power of the microscope and of the telescope.

70. The photographic camera consists of a light-tight box, in the front of which a lens is mounted. At the back of this box is a sensitive plate, upon which an image of external objects is projected by the lens. The various types of highly perfected photographic lens systems are described in Chapter VI.

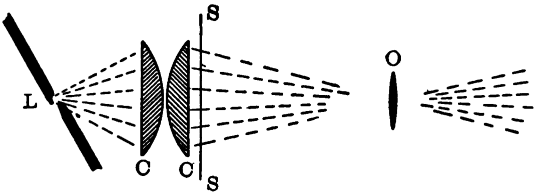

Fig. 69.

71. The projection lantern is an arrangement for projecting the image of a brilliantly illuminated object or picture upon a screen. The light from a brilliant lamp *L*, Fig. 69, passes through *condensing* lenses *CC*, through a transparent picture *SS* and through a lens *O*, which focusses the light from each point of *SS* at a corresponding point upon the screen, thus giving an enlarged image of *SS* on the screen. The lens *O* is called the lantern *objective*. In practice the lantern objective consists of a compensated lens system as explained in Chapter VI., Art. 92.

72. The simple microscope. Definition of magnifying power.— The simple microscope, or *magnifying glass*, consists of a converging lens. This lens is held as near to the eye as possible and the object to be examined is moved up until it is seen most sharply. The eye is then looking at an enlarged virtual image of the object, and this virtual image is at the distance of most distinct vision from the eye, say, 25 centimeters.

The magnifying power of a microscope is defined as the quotient of the apparent size (or visual angle) of an object as seen with the microscope, divided by the apparent size (or visual angle) of the object as seen with the naked eye at a distance of 25 centimeters.

73. Proposition.—The magnifying power of a magnifying glass is:

$$m = \frac{25}{p} + 1 \qquad (9)$$

in which p is the focal length of the magnifying glass in centimeters. The eye is supposed to be accommodated for a distance of 25 centimeters.

Proof.—Let *O*, Fig. 70, be the object and *i* its virtual image formed by the magnifying glass. The eye being accommodated for 25 centimeters, the object will be moved until the image *i* is 25 cm. from the eye. The distance from the lens to the eye may be neglected, so that the distance *b* of the image from the lens is 25 cm. Since the distance from a lens to a virtual image is considered as negative, we have, from equation (8), Art. 55:

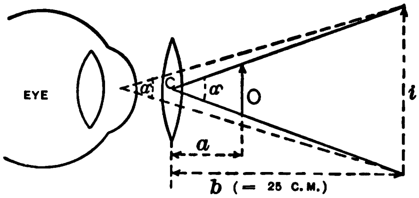

Fig. 70.

$$\frac{1}{p} = \frac{1}{a} - \frac{1}{25} \qquad (a)$$

whence

$$a = \frac{25p}{25 + p} \qquad (b)$$

Now, the visual angle of the image *i* is the same as the angle at *C* subtended by the object *O* since the distance from the magnifying glass to the eye is neglected. *This angle, a, is as many times as great as the angle that O would subtend at a distance of 25 centimeters, as 25 is greater than a, Fig. 70 ;* that is

$$m = \frac{25}{a} \qquad (c)$$

Substituting the value of a from equation (b) in equation (c) we have equation (9).

Remark 1. — When the eye is accommodated for parallel rays b, Fig. 70, is equal to infinity, a is equal to p, and m is equal to $25/p$.

Remark 2. — A high-power magnifying glass must have very short focal length, and it must be held very near to the object to be examined. Furthermore a short focal length lens is necessarily small in diameter and the field of view in such a magnifying glass is extremely limited. Therefore high-power magnifying glasses are inconvenient and unsatisfactory. For magnifying powers exceeding 20 or 30 diameters the *compound microscope* is to be preferred.

74. The compound microscope consists of a lens A, Fig. 71, which forms an enlarged real image i of an object, and a magnifying glass B for viewing this image. The lens A is called the *object glass* or *objective* and the lens B is called the *eye-piece.* In modern high-grade microscopes the objectives and eye-pieces consist of compensated lens systems as explained in Chapter VI.

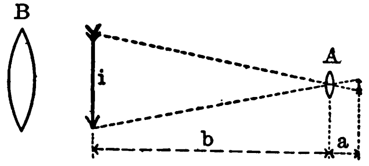

Fig. 71.

Remark. — The dotted lines in Fig. 71 are not intended to represent rays but only to show geometrical relations.

75. Proposition. — The magnifying power of a compound microscope is:

$$m = \frac{b}{a}\left(\frac{25}{p} + 1\right) \qquad (10)$$

in which a and b are the respective distances of object and image from the center of the object glass, as shown in Fig. 71, and p is the focal length of the eye-piece.

Proof. — The image is b/a times as large as the object, and the eye-piece, being a magnifying glass, makes this image appear

$(25/p + 1)$ times as large as it would appear to the naked eye at a distance of 25 centimeters, or $b/a\ (25/p + 1)$ times as large as the object would appear at a distance of 25 centimeters from the naked eye.

Remark. — The greatest effective magnifying power even of an optically perfect compound microscope is about 900 diameters. This matter is explained in Chapter VI., Art. 82.

76. The telescope consists of a long focus lens O, Fig. 72, which forms an image i of a distant object and a magnifying glass E for viewing this image. The lens O is called the *object glass* or *objective* and the lens E is called the *eye-piece*. In modern high-grade telescopes the objectives and eye-pieces consist of compensated lens systems, as explained in Chapter VI.

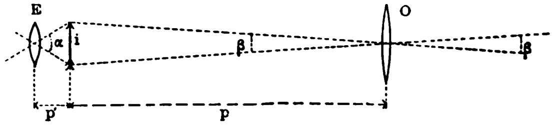

Fig. 72.

Remark. — The dotted lines in Figs. 72 and 73 are not intended to represent rays but only to show geometrical relations.

77. The magnifying power of a telescope is defined as the quotient of the visual angle of a distant object as seen with the telescope divided by the visual angle of the object as seen with the naked eye. The visual angle of the object as seen with the naked eye is the same as the angle β, Fig. 72, which the object subtends at the center of the object glass; and the visual angle which the object subtends as seen through the telescope is the angle α which the image i subtends at the center of the lens E (see Art. 73). Therefore the magnifying power of the telescope is

$$m = \frac{\alpha}{\beta} \qquad (a)$$

If we consider the object to be, sensibly, at an infinite distance, then i is at the principal focus of O and the distance Oi is the focal length p of O. If the eye is accommodated for parallel rays then i is at the principal focus of E and the distance Ei is the focal length, p', of E. Therefore, since both angles α and β are supposed to be quite small, we have

$$\frac{\alpha}{\beta} = \frac{p}{p'} \tag{b}$$

and from equations (a) and (b) we have

$$m = \frac{p}{p'} \tag{11}$$

That is, the magnifying power of a telescope is equal to the ratio of the focal length of the objective to the focal length of the eye-piece.

Remark. — There is no optical limit to the effective magnifying power of the telescope, as there is to that of the microscope. The longer and larger the telescope the greater its effective magnifying power may be. Compare Art. 75.

78. The erecting telescope or spy-glass. — The simple telescope shows objects inverted. The spy-glass is a telescope modified so as to make distant objects appear erect. Fig. 73 shows the essential parts of a spy-glass. The objective lens O forms at i an inverted image of a distant object. The lens S forms at i' an inverted image of i, or an erect image of the distant object. This erect image i' is viewed by a magnifying glass E. The introduction of the erecting lens S lengthens the telescope very materially and the spy-glass is consequently very long in comparison with its diameter. In practice the lenses O, S, and E, Fig. 73, consist each of a compensated lens system.

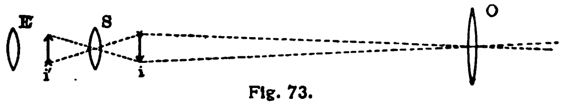

Fig. 73.

79. The opera-glass is a telescope of which the eye-piece is a diverging lens. The action of the opera-glass is shown in Fig.

74, in which both images *i* and *i'* are virtual images and therefore represented by dotted lines. The image of a distant object formed by the object glass *O* would be at *i* were it not for the interposition of the lens *E*. The effect of the lens *E* is to cause the pencil of rays which is converging towards each point of *i* to appear to have come from corresponding points of *i'* so that *i'* is a virtual image of *i* formed by the eye-piece *E*, and the observer in looking through the instrument sees this image *i'* which is of course erect. This telescope is very short for a given magnifying power.

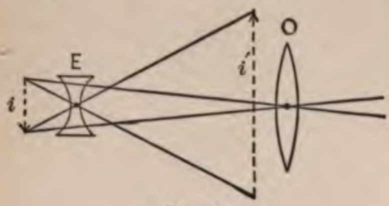

Fig. 74.

80. Use of the telescope for sighting. — Sighting cannot be accurately done where the two sights consist of pins or point-like projections as on a gun. The most accurate sighting that can be done with the naked eye in this way is perhaps with an error of one or two minutes of angle. Accurate sighting is always done with the telescope. Two very fine wires are stretched across the focal plane *pp* of the object glass of the telescope, as shown in Fig. 75. Fig. 76 shows a view of these cross wires as they appear to an observer looking through the eye-piece. These cross wires are in the same plane as the image of the distant object which is viewed by the telescope; these

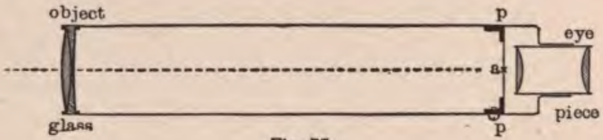

Fig. 75.

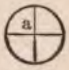

Fig. 76.

wires are therefore sharply in focus, and the telescope may be adjusted so as to bring the image of any given point of the object accurately coincident with the point *a* where the two cross wires intersect. *Then the line drawn from a through the center of the object glass* * *passes through the given point of the object.*

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[*] Strictly, the line drawn from *a* through the posterior nodal point of the objective lens system, is parallel to the line drawn from the anterior nodal point of the system to the given point in the object.

CHAPTER VI.

LENS IMPERFECTIONS AND THEIR COMPENSATION.

81. General statement. — The elementary theory of lenses given in Chapter IV., applies to thin lenses and to centered lens systems of small aperture. It assumes all imaged points to be nearly in the axis of the lens or lens system ; and it ignores the variation of focal length of a lens with the wave-length (color) of the light. This elementary theory of lenses is an approximate theory and an actual lens is always more or less imperfect; that is, an actual lens does not bring light accurately to a focus and the image formed by a lens is always more or less distorted and blurred. In the following articles the various imperfections of lenses are discussed and the methods for compensating these imperfections are described.

It is interesting to note in this connection the tendency among astronomers during the latter part of the seventeenth century in their attempts to increase the power of their telescopes. It was found necessary, since simple lenses were used, to increase the length of the telescope enormously so that the object glass might be a *thin lens* with very slightly curved surfaces. In some cases telescopes were made fifty meters or more in length. The object glass was pivoted on top of a tall pole,* and the eyepiece was mounted on a tripod on the ground; the connecting tube, which is a prominent feature of modern large telescopes, being omitted.

82. Numerical aperture. — The free diameter of a lens divided by its focal length is called the *numerical aperture* of the lens.

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* See Hastings' "Light," p. 57 (Scribner's Sons, 1901).

Thus telescope objectives range up to $\frac{1}{10}$, photographic objectives up to $\frac{1}{8}$, and microscope objectives up to 1.4 numerical aperture.

Large numerical aperture is important for two reasons, as follows:

(*a*) The *brightness of the image* which a lens forms of a given distant object is proportional to the square of the numerical aperture of the lens. Thus if a lens of given focal length has its free diameter doubled (numerical aperture doubled), it will admit four times as much light and the image will be four times as bright. If a lens A has a focal length twice as great as the focal length of a lens B, then the image formed by A of a given distant object will be twice as large in linear dimensions or four times as large in area as the image formed by B, and if the lens A has a free diameter twice as large as the free diameter of B (same numerical aperture) it will admit four times as much light which, being spread over an image four times as large in area, will give an image of the same brightness as B.

(*b*) *Resolving power.* — The power of a lens to reproduce fineness of detail of an object in an image is called the *resolving power* of the lens. If a lens concentrated the light from the various points of an object at actual points in the image, then infinite fineness of detail would exist in the image and the lens would have unlimited resolving power. However, a lens concentrates light from a point at a *spot*, partly on account of imperfections of the lens and partly on account of the essential nature of light itself. The imperfections of a lens may be practically eliminated by compensation, as described in this chapter. The limited resolving power of a completely compensated lens depends upon the nature of light, and a compensated lens must have the largest possible numerical aperture in order to give the greatest possible resolving power. The dependence of resolving power upon numerical aperture is explained as follows:

Consider a large sector of a spherical wave converging towards its center of curvature F, as shown in Fig. 77, and a small sector of a spherical wave converging towards its center of curvature *F*, as shown in Fig. 78. The large spherical wave in Fig. 77 may come, for example, from a compensated lens of large numerical aperture, and the small wave in Fig. 78 from a compensated lens of small numerical aperture. The entire disturbance which is involved in the converging wave, Fig. 77 or Fig. 78, is *not* concentrated at the point *F*, but a very considerable portion of this disturbance leaks off, as it were, from the free ends of the wave *ccc* into the regions *RR* and this disturbance which leaks off is largely concentrated near *F* so that the focus is a spot instead of a point. The leakage, if we may so name it, is a much larger part of the whole disturbance in Fig. 78 than it is in Fig. 77, so that the focus of a lens of large numerical aperture is sharper than the focus of a lens of small numerical aperture.

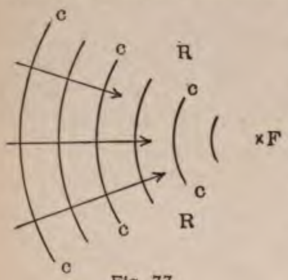


Fig. 77.

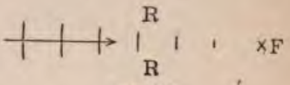


Fig. 78.

The largest numerical aperture that is feasible is about 1.4, as exemplified in the high-grade objectives of modern microscopes. With a perfect objective of this numerical aperture the *spot* in the image, corresponding to a point in the object, would be barely visible as a spot of perceptible size with a magnifying power of 900 diameters.* Therefore any increase of the magnifying power of a microscope beyond 900 diameters is ineffective inasmuch as such increase of magnifying power cannot bring out additional detail. Thus, a mosaic picture shows more and more detail as one approaches it from a great distance until the separate stones become visible as spots, after which no additional detail is possible.

The phenomenon here described as the "leaking off" of the disturbance from the free ends of a wave *ccc*, Fig. 77, is called diffraction. This phenomenon is discussed in Chapter VIII.

* See Drude, "Lehrbuch der Optik," p. 218.

83. Field angle. — The angle at the center of a lens, between lines drawn to the extreme edges of the largest distinct image which the lens can produce, is called the *field angle* of the lens. Photographic objectives are made which give excellent definition with a field angle as great as 110°. Such lenses are called *wide-angle* lenses. The field angle of telescope and microscope objectives is ordinarily very small, seldom exceeding four or five degrees.

84. Spherical aberration. — When a plane or spherical wave passes through a simple lens of wide aperture the transmitted wave is not spherical, and, therefore, not concentrated at a point. In fact, the portions of the wave which pass through the edge zones of the lens are focussed nearer the lens than the portions of the wave which pass through the central zone of the lens, as shown in Fig. 79. This fault of a lens is called *spherical aberration*.

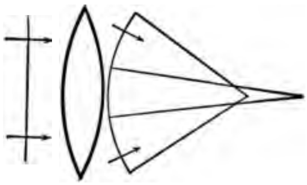


Fig. 79.

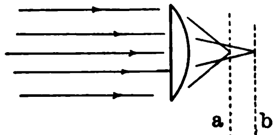
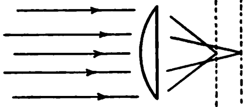
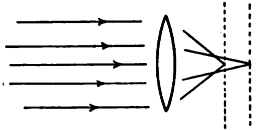
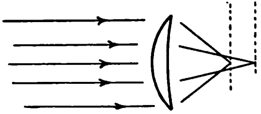
The difference in focal length of edge and central zones of a lens of given diameter and given focal length varies with the relative curvature of the two surfaces of the lens, and is much less for glass of high refractive index than it is for glass of low refractive index, as may be seen from the table on next page.

Form No. 4 has the least spherical aberration for parallel rays of any simple lens, for given value of μ. The radii of curvature of the two surfaces are as 1 : 6.

70. Compensation of spherical aberration. Aplanatism. — A converging lens and a diverging lens of equal and opposite focal lengths and equal and opposite spherical aberration entirely annul each other's action when they are placed near together and used as a single lens. It is possible, however, to make a converging lens and a diverging lens of unequal focal lengths but of equal and opposite spherical aberration. When these two lenses are

TABLE OF SPHERICAL ABERRATIONS.

Focal length of lens, 100 cm. Diameter of Lens, 10 cm.

	Form of lens.	$\mu = 1.5$	$\mu = 2.0$
1		$ab = 4.5$ cm.	$ab = 2.0$ cm.
2		$ab = 1.17$ cm.	$ab = 0.5$ cm.
3		$ab = 1.67$ cm.	$ab = 1.0$ cm.
4		$ab = 1.07$ cm.	$ab = 0.44$ cm.

combined and used as a single lens, spherical aberration is annulled but the converging action of the one lens is not annulled by the diverging action of the other. Such a lens system is free from spherical aberration and is said to be *aplanatic*. A lens system can be accurately aplanatic for a given pair of conjugate points only. These conjugate points are called the *aplanatic points* of the system.

Abbe's sine law. — A lens may be aplanatic for a pair of conjugate points lying in the axis of the lens and yet be distinctly non-aplanatic for a pair of conjugate points lying in a line only very slightly inclined to the axis of the lens. Such a lens would give a sharp image only at the very center of its field. If a lens is to form a sharp image of a small group of points near its axis the lens must be aplanatic for points near its axis as well as for its so-called aplanatic points in the axis. The condition that must be satisfied in order that this state of affairs may be realized is called from its discoverer *Abbe's sine law*.

Preliminary statement. — Let *a* and *b*, Fig. 80, be two conjugate points with respect to a lens system. A spherical wave passing out from *a* becomes a spherical wave coming in upon b. Let w and w' be the portions of this spherical wave which have traveled along any two rays r and r'. Since w and w' start out from a at the same instant, and are portions of a spherical surface with its center at b, they will reach b at the same instant. It follows, therefore, that *the time required for a light wave to travel along any two rays from a point to its conjugate is the same.* This time is also the least time in which light can pass from one point to the other. This principle is sometimes the called *principle of least time.*

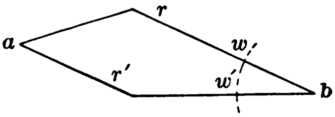


Fig. 80.

Consider a *very small* object aa' and its image bb', Fig. 81. Let d be the distance aa' and md the distance bb', m being the magnification. Consider the rays R (the axis) and R' from a to b, and the rays r and r' from a' to b'. Also consider the two sensibly equal angles ϕ, and the two sensibly equal angles Φ as shown, remembering that aa' is very short and that bb' is very short.

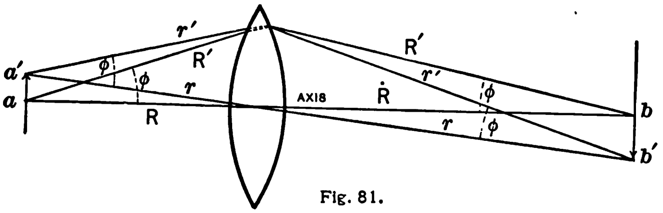


Fig. 81.

The rays R and r pass through the same portion of the lens, and are of the same length, so that the rays r' and R' must be the same (optical) length also. The rays r' and R' pass through the same thickness of glass, the incident portion of R' is $d \sin \phi$ longer than the incident portion of r', and the emergent portion of r' is $md \sin \Phi$ longer than the emergent portion of R', as shown in Fig. 82. Therefore, $d \sin \phi = md \sin \Phi$, or

$$\frac{\sin \phi}{\sin \Phi} = m \qquad (a)$$

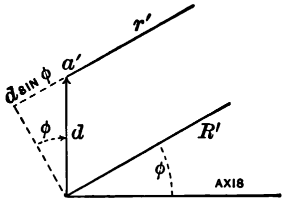


Fig. 82.

If the refractive indices of the media in which the object and image are located are μ and μ' respectively, then the optical values of $d \sin \phi$ and $md \sin \Phi$ are $md/\mu \cdot \sin \phi$ and $md/\mu' \cdot \sin \Phi$ respectively, and equation (a) becomes

$$\frac{\sin \phi}{\sin \Phi} = m\frac{\mu}{\mu'} \qquad (b)$$

The reader will find, by applying equation (b), that the sphere described in Art. 49 satisfies Abbe's sine law, and is therefore aplanatic for all points near the axis in a plane passing through the point P, Fig. 45, perpendicular to the axis. If this were not the case, a microscope provided with the objective, shown in Fig. 102, would give sharp definition only in the very central point of the field of view.

86. Chromatic aberration. — The value of the refractive index of a given sample of glass varies with the wave-length of the light. Therefore the focal length of a lens is different for different wave-lengths of light, that is for different colors. Thus violet light is focussed nearer the lens than red light as shown at v and r, Fig. 83. This variation of focal length of a lens with the wave-length of light is called *chromatic aberration.*

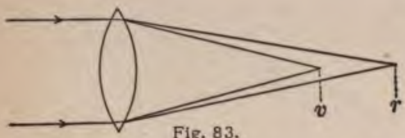


Fig. 83.

The distance vr, Fig. 83, for a lens of given focal length varies greatly with different kinds of glass and it is therefore possible to construct a converging lens and a diverging lens of different kinds of glass so that when combined and used as a single lens the chromatic aberration of the one is annulled by the opposite chromatic aberration of the other while the converging power of the one is not annulled by the diverging power of the other. Such a combination is called an *achromatic doublet.* A sketch of the theory of the achromatic doublet is given in the chapter on Dispersion. The achromatic doublet has the same focal length for only two definite colors, say red and blue or red and violet, as is explained in Chapter VII.

A lens system consisting of two lenses of similar glass at a distance apart equal to half the sum of their individual focal lengths has the same focal length for all wave-lengths. The doublets of Ramsden and of Huygens (see Art. 92) so much used for eye-pieces of telescopes and microscopes are examples of achromatic systems of this type.

A careful study of articles 62 to 67 will show that the errors of a lens system due to chromatic aberration are not completely eliminated when the system is made to have the same focal point F, Fig. 61, for all colors, but that in addition the principal planes PP' of the system must have the same positions for all colors. That is to say the *position of the focal point* F and the *focal length* PF have both to be made the same for all colors or for two colors. The effect of achromatization of the focal point

F with non-achromatization of the principal planes *PP′* is to produce in the focal plane *F* colored images of *different sizes* of an infinitely distant object.

The telescope objective (see Art. 92) is so thin a lens system that the principal planes of the system are practically coincident with the central plane of the system and achromatization of principal planes is meaningless. Achromatization of focal length is alone sufficient and complete.

The doublets of Ramsden and Huygens are achromatized for focal length but not for position of focal points (or principal planes). For complete achromatization these doublets must have each of their elementary lenses replaced by an achromatic doublet like the telescope objective.*

87. Astigmatism. — A pencil of parallel rays, in fact any homocentric pencil of rays, becomes an astigmatic pencil when it passes obliquely through a simple lens. Thus Fig. 84 shows the positions of the focal lines *C* and *DD* of the astigmatic pencil of rays produced by the passage of the pencil of parallel rays *rr* obliquely through the lens as shown. The focal point of the lens is at *F*. This conversion of homocentric pencil into an astigmatic pencil by oblique passage through a lens is called *astigmatism*.

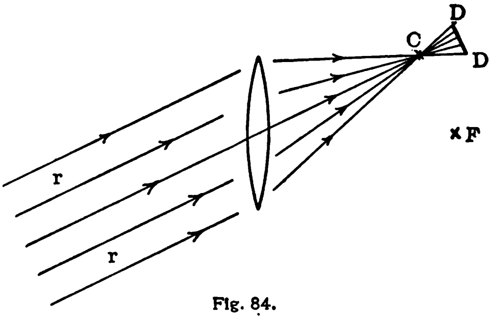


Fig. 84.

A lens system which is free or approximately free from astigmatism is called an *anastigmatic* or simply a *stigmatic* system. Such lens systems have been designed by P. Rudolph and others.

There is no general theorem concerning the possibility of compensating the astigmatism of one lens by the opposite astigmatism of another lens as there is in case of spherical aberration and chromatic aberration. In fact, the design of a stigmatic lens system is accomplished by actual trial helped more or less by detailed calculations by trigonometric methods of the passage of narrow pencils of rays through various proposed combinations of lenses.

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* For further details see Drude, "Lehrbuch der Optik," p. 62–67.

A lens held obliquely as shown in Fig. 85 can form a fairly sharp image of a line AB *provided the line* AB *is parallel to the one or the other of the focal lines* C *(perpendicular to paper) or* DD. If the line AB is parallel to DD as shown in the figure, then, in order that a sharp image of the line may be formed on the screen, the focal line DD must be in the plane of the screen. If the line AB is parallel to the focal line C, then, in order that a sharp image may be formed on the screen, the focal line C must be in the plane of the screen. This may be strikingly shown by projecting the image of a cross-ruled lantern slide by means of a projection lantern, the objective lens of the lantern being held obliquely as shown in Fig. 85. In this experiment the central zone only of the projecting lens should be used. Under these conditions either set of parallel rulings on the cross-ruled slide may be sharply focussed on the screen.

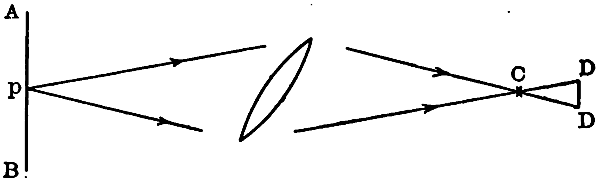

Fig. 85.

Astigmatism of the eye is due to unequal curvatures of the refracting surfaces of the eye in different directions so that a homocentric pencil of rays entering the eye *axially* becomes an astigmatic pencil. Astigmatism of the eye is shown by one's inability to see both vertical and horizontal lines distinctly; it is corrected by using eye glasses with *cylindrical* instead of spherical surfaces, thus compensating for the inequalities of curvature of the refracting surfaces in the eye.

88. Distortion. — The image of an object formed by a lens is in general distorted. This distortion is due to increase or decrease of magnification with increasing distance from the axis in the focal plane in which the image is formed. In case the magnification is uniform over the entire field there is no distortion and the image of a square network will appear as shown in Fig. 86. In case the magnification *increases* towards the edge of the field the image of a square network will appear as shown in Fig. 87. In case the magnification *decreases* towards the edge of the field the image of a square network will appear as shown in Fig. 88. A lens system which gives an undistorted image of an object is called an *orthoscopic* or *rectilinear* system. The sim-

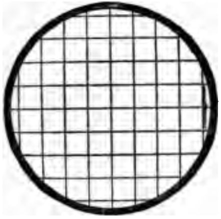

Fig. 86.

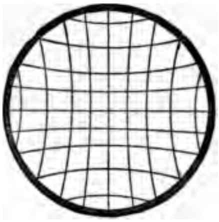

Fig. 87.

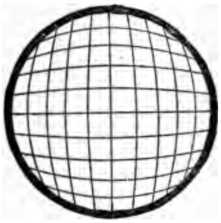

Fig. 88.

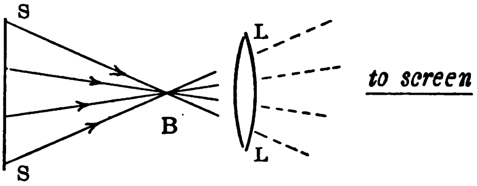

Fig. 89.

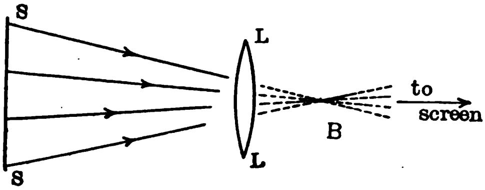

Fig 90.

plest orthoscopic system is the *symmetrical doublet* first introduced by Steinheil. This doublet is shown in Fig. 91. It consists of two similar lenses placed at equal distances on the two sides of a diaphragm having a small hole at its center.

The distortion of image by a lens may be beautifully shown as follows: A projection lantern is provided with a simple objective lens *LL*, Figs. 89 and 90, and the image of a cross-ruled lantern slide *SS* is projected upon the screen. The source of light in the lantern is adjusted so that the light after passing through *SS* is concentrated at *B*, Fig. 89, *as if the point B were a small hole in a diaphragm in front of the objective lens LL.* In this case the cross-rulings as projected on the screen appear like Fig. 88. The source of light in the lantern is then adjusted so that the light after passing through *SS* is concentrated at *B*, Fig. 90, *as if the point B were a small hole in a diaphragm behind the objective lens LL.* In this case the cross-rulings as projected on the screen appear like Fig. 87.

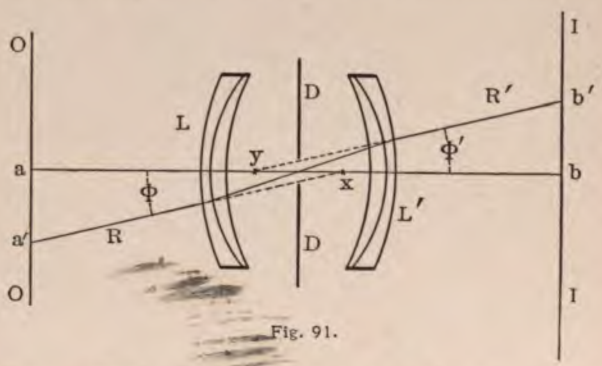

Fig. 91.

This experiment shows that the distortion of images is of one kind or the other according as the diaphragm *B* is in front of or behind the lens and one is able to see in a general way that the one kind of distortion is compensated by the other when the diaphragm is between two lenses of a system as shown in Fig. 91; that is to say, when the diaphragm is *in front of* one lens and *behind* the other lens.

Theory of the orthoscopic lens. — Two similar achromatic doublets L and L', Fig. 91, are placed at equal distances from a diaphragm DD with a small hole at its center as shown in the figure. OO is the plane of the object and II is the plane of the image.

The lens L will cause any ray R which is directed towards a certain point x to actually pass through the center, H, of the hole in the diaphragm, the points H and x being conjugates with reference to the lens L. Similarly, any ray R' which has actually come from the point H will appear to come from a certain point y after passing through the lens L', the points H and y being conjugates with reference to the lens L'.

The ray RR' connects the object point a' with the corresponding image point b' and if the ratio of the distance aa' to the distance bb' is *constant* wherever the points a' and b' may be in the planes OO and II, then the image is geometrically similar to the object and there is no distortion. From the figure we have $aa' = aH \times \tan\phi$ and $bb' = Hb \times \tan\phi'$ so that

$$\frac{aa'}{bb'} = \frac{aH}{Hb} \times \frac{\tan\phi}{\tan\phi'}$$

Now, the distances aH and Hb are constant, and the constancy of the ratio aa'/bb' requires the condition

$$\frac{\tan\phi}{\tan\phi'} = \text{constant}$$

This equation expresses the condition which must be satisfied in order that the lens system may be orthoscopic. In case of the symmetrical doublet, which is the case actually represented in Fig. 91, the two angles ϕ and ϕ' are *equal* so that, of course, the ratio $\tan\phi/\tan\phi'$ is a constant.

89. Curvature of field. — In order to project upon a screen the most distinct image of an extended object which it is possible to form by means of a simple lens, the screen must be curved, as shown by SS, Fig. 92. It is instructive to note the connection between Figs. 84 and 92. There is, between C and DD, Fig. 84, a place where the astigmatic pencil would produce the smallest possible luminous spot, the circle of least confusion, on a screen (see Art. 28), and the screen SS, Fig. 92, must pass between C and DD of Fig. 84.

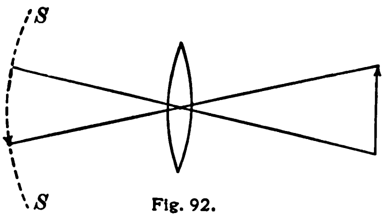

Fig. 92.

The imperfection of a lens here described is called *curvature of f* [illegible]re of the field of a simple lens of given focal length depends upon the refractive index of the glass and it is possible * to combine a converging lens of one kind of glass with a diverging lens of another kind of glass so as to form a doublet of any desired focal length which will have a *flat field*. Flatness of field is very important in photographic objectives inasmuch as such lenses are used to project images upon flat photographic plates.

90. Wide angle lenses vs. wide aperture lenses.—A lens cannot be made to give a wide field angle and to have at the same time a large numerical aperture; the two things are incompatible. When a very wide field angle is desired one must be content with small aperture, and when a very wide aperture is desired one must be content with small field angle. The largest feasible numerical aperture is about 1.4 with a field angle of, say, five degrees, as exemplified in the high-grade microscope objective. The largest feasible field angle, with a flat field, is about 110° with a numerical aperture of about $\frac{1}{86}$, as exemplified in the so-called wide-angle photographic objective.

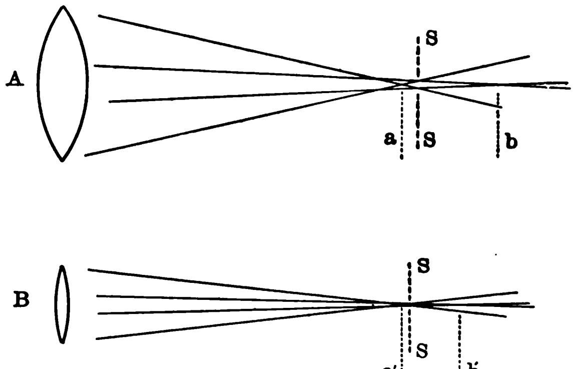

Fig. 93*a*.

There is a demand in photography for lenses of medium wide aperture giving a medium wide field, and lenses are made giving good definition over a field from 40° to 60° wide with numerical aperture ranging from $\frac{1}{4}$ to $\frac{1}{6}$, as exemplified by the Zeiss photographic objectives.

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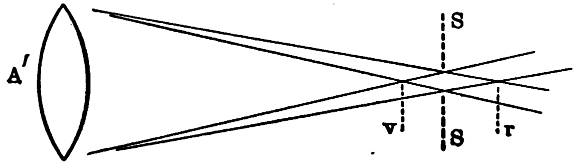
* See Lummer, *Zeitschrift für Instrumentenkunde*, 1897, p. 231.

A wide aperture lens must be, above all things, compensated for spherical aberration and for chromatic aberration. The effect of spherical aberration in producing very great blurring at the focus of a large aperture lens is shown in Fig. 93*a*. The same effect for chromatic aberration is shown in Fig. 93*b*. The distance *ab*, Fig. 93*a*, is roughly proportional to the square of the aperture of the lens, so that *a′b′* is very much less than *ab*. The dotted line *SS* represents the position of a screen upon which the lenses *A* and *B* would most sharply focus a beam of light and Fig. 93*a* shows that the light is focussed in a very



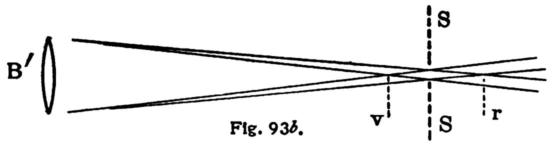


Fig. 93*b*.

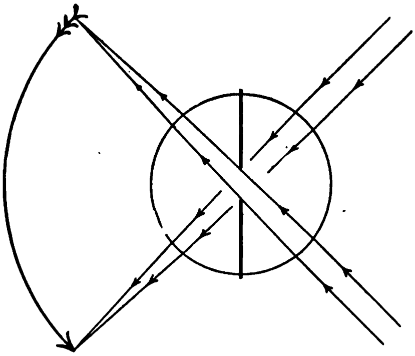


Fig. 94*a*.

much smaller spot by lens B than by lens A. The distance rv, Fig. 93*b*, is independent of the aperture of the lens and this figure shows that a beam of light is focussed in a much smaller spot by lens B' than by lens A'.

A wide-angle lens must be, above all things, compensated for astigmatism, distortion and curvature of field.

91. The wide-angle lens.—A lens in the form of a complete sphere, having a diaphragm with a small hole at the center of the sphere as shown in Fig. 94*a*, has a field angle of nearly 180°, but the field is very strongly curved. All wide-angle photographic objectives involve the principle of the spherical lens, modified more or less to give achromatism and flatness of field. Fig. 94*b* is a sectional view of an achromatic orthoscopic wide-angle photographic objective.

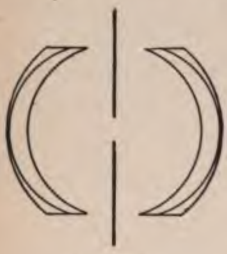


Fig. 94*b*.

92. Examples of compensated lens system.—(*a*) *Telescopic objectives* are used for very small field angle and they are therefore compensated only for spherical aberration and chromatic aberration. The achromatization is accomplished by using a converging lens of crown glass and a diverging lens of flint glass as explained in Art. 94. The condition which must be satisfied to produce achromatism [see equation (12), Art. 94] leaves some freedom of choice as to the relative curvatures of the different surfaces of the two lenses, and these curvatures are chosen so that the spherical aberration of the converging crown lens will be compensated by the opposite spherical aberration of the diverging flint lens. Fig. 95 shows a sectional view of an achromatic aplanatic telescope objective. The possibility of compensating for chromatic aberration was discovered by Dolland in 1758 and shortly afterwards the further possibility of compensating for spherical aberration was discovered.

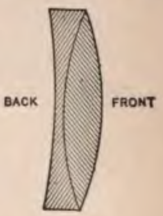


Fig. 95.

(*b*) *Magnifying glasses and eye-pieces.* — Magnifying glasses and eye-pieces are used under conditions which involve small numerical aperture because of the delimiting action of the pupil of the observer's eye; that is to say, the pencil of rays which enters the observer's eye from a given point of the object passes through a very small portion, only, of the lens as shown in Fig. 96.

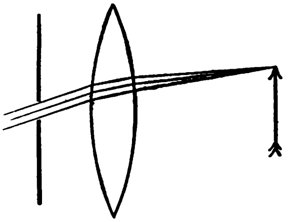


Fig. 96.

On the other hand, the conditions of use of magnifying glasses and eye-pieces involve comparatively wide field angles, in some cases as great as 50° or more. Fig. 97 shows an eye-piece doublet designed by Ramsden in 1783. It consists of two similar plano-convex lenses of similar glass placed somewhat nearer together than half the sum of the two focal lengths. This doublet is partially achromatic,* and it has a fairly wide flat field. It is much used in modern microscopes and telescopes. Fig. 98 shows an eye-piece doublet designed by Huygens about 1680. It consists of two lenses of similar glass, the respective focal lengths are as 3 : 1, and they are placed at a distance apart equal to half the sum of their respective focal lengths. This doublet is partially achromatic,* and it has a very wide flat field. It is much used in modern microscopes and telescopes.

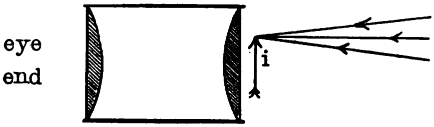


Fig. 97.

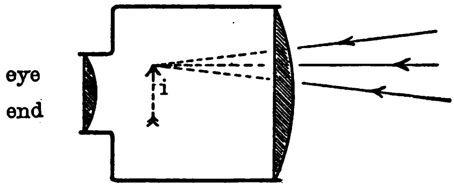


Fig. 98.

Remark. — The arrow *i* in Fig. 97 shows the position of the image (formed by the object glass of the microscope or telescope) which is being examined by the Ramsden eye-piece. This is a real image and cross wires may be placed in the plane of this image *i* and be seen distinctly by the observer. The arrow *i* in Fig. 98 shows the position of the image (formed by the object glass of the microscope or telescope) which is being examined by the Huygens eye-piece. This is a virtual image and cross wires cannot be placed in the plane of this image and be distinctly seen by the observer. Therefore the Ramsden eye-piece is employed in most telescopes and microscopes where cross wires are used. Compare Fig. 75, Art. 80. The large lens of the Huygens eye-piece forms, however, a real image of *i*, Fig. 98, slightly nearer to the large lens than the virtual image *i* itself and cross wires are sometimes placed in the plane of this real image.

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* Achromatized for focal length only and not for positions of principal planes.

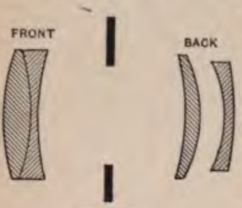

Fig. 99.

(*c*) *Photographic lenses.** — For photographic portraiture very quick acting (wide aperture) lenses are necessary. Fig. 99 shows a portrait lens by Dallmeyer with a numerical aperture of about ¼. For group photography, wideness of aperture has to be more or less sacrificed to wideness of field. Fig. 100 shows a Zeiss anastigmatic lens, designed by Rudolph, which gives a 50° field with a numerical aperture of ⅙.

For the copying of drawings and for architectural photography, a lens must be orthoscopic and in some cases very wide angle lenses are required. Fig. 91 shows the standard form of symmetrical orthoscopic lens, and Fig. 94*b* shows a very wide angle lens which is also orthoscopic.

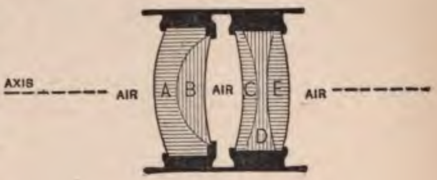

Fig. 100.

For landscape photography the medium or wide-angle orthoscopic lens is ordinarily used.

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* A good outline of photographic optics is given by Lummer in the *Zeitschrift für Instrumentenkunde*, 1897, pages 208, 225 and 264.

The objective lens of a projection lantern is usually a symmetrical orthoscopic lens like Fig. 91.

Remark. — Each half of a symmetrical orthoscopic lens is achromatized in very much the same way as the simple achromatic telescope objective.

(*d*) *Microscope objectives.* — Fig. 101 shows a highly perfected microscope objective designed by Abbe about 1885, and known as the *apochromatic* objective. It is designed to give the largest possible numerical aperture for the sake of resolving power, and although corrected only for spherical aberration and for chromatic aberration, it involves eleven separate compensating effects. It is achromatized for *focal length* for three colors and its spherical aberration is eliminated for each of two colors for points out of the axis as well as for points in the axis. The lack of achromatization of the *principal planes* causes the lens to form colored images of different sizes which imperfection is compensated by using a specially designed non-achromatic eye-piece which magnifies the differently colored images differently, thus giving a single colorless resultant image. The limit of effective magnifying power of a microscope using perfect lenses has been mentioned in Art. 82. Abbe has very nearly reached this limit and but little remains to be done in the development of the microscope.

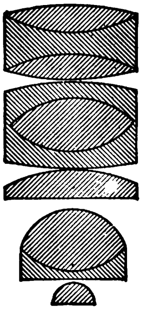

Fig. 101.

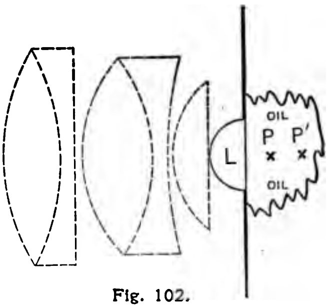

Fig. 102.

The highest grade of microscope objective, having the largest attainable numerical aperture, works with what is called *homogeneous immersion* of the object which is being viewed. The purpose of this method of working is to utilize to the utmost the one case in which the spherical refracting surface is aplanatic. This case, as described in Art. 49, requires the light which is to be refracted to come from a luminous point inside of the refracting substance. Fig. 102 shows the arrangement of a microscope objective for homogeneous immersion of object. The object *P* which is being examined by the microscope is submerged in cedar oil which has the same refractive index as the glass of the front lens *L* of the objective. This oil entirely fills the space between *P* and *L* and the point *P* is, in effect, inside of the refracting substance of the lens *L*.

CHAPTER VII.

DISPERSION. SPECTRUM ANALYSIS.

93. Newton's experiment. *Homogeneous light. Non-homogeneous light.* — A beam of parallel rays of white light, such as sunlight or lamplight, is changed into a fan-like beam by passage through a prism. Thus the beam of parallel rays *B*, Fig. 103, is changed into the fan-like beam *B'* by passage through the prism *P*. This fan-like beam falling upon a screen *SS* produces an illuminated band *RV*, called a *spectrum*, which is red at the end *R* and passes by insensible gradations through orange, yellow, green, and blue to violet at the end *V*. The beam *B* of white light is said to be *dispersed* by the prism. A photographic plate reveals the existence of invisible rays beyond *V*. These rays are called *ultra-violet* rays. The brilliant light of the sun and of the electric arc are especially rich in ultra-violet rays. A thermopile or bolometer (Art. 166) shows the existence of rays below *R*, Fig. 103. These rays are called *infra-red* rays. The portion of the spectrum between *R* and *V* is called the *visible spectrum.*

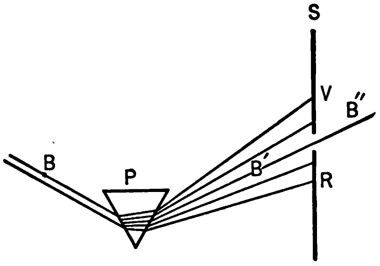

Fig. 103.

The original beam of white light *B*, Fig. 103, is deflected by the prism and also spread out or dispersed. The beam of light *B''*, passing through a small hole in the screen, can be deflected by a prism but it cannot be spread out or dispersed. The beam *B''* consists of one kind of light only and it is therefore said to be *homogeneous*.* The beam of white light *B* is composed of many different kinds of light and it is, therefore, said to be *non-homogeneous*. Since the prism P deflects the different portions of the non-homogeneous beam B differently, it is obvious that the glass of which the prism is composed has a different refractive index for each of the homogeneous parts or components of white light.

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* Homogeneous light is sometimes called *monochromatic* light.

The phenomena of interference (see Chapter VIII.) show that a beam of homogeneous or monochromatic light is a *simple wave-train of definite wave-length.** A beam of non-homogeneous light consists of a number, sometimes an infinite number, of distinct simple wave-trains of different wave-lengths. In vacuum these different wave-trains travel at the same velocity, and this is approximately true in air also. In dense substances like glass wave-trains of different wave-lengths have distinctly different velocities, and when a beam of non-homogeneous light enters glass the different wave-trains are differently refracted; that is the wave-trains of different wave-lengths are dispersed.

94. The achromatic lens.—A prism of flint glass which gives a spectrum of the same length, say from red to blue, as a prism of crown glass, produces, on the whole, much less total deflection of the beam than does the crown-glass prism. If, therefore, the two prisms are placed together as shown in Fig. 104, the dispersion of the crown prism will be annulled by the dispersion of the flint prism, but the deflection will not be wholly annulled. The compound prism therefore gives deflection without dispersion. However, spectra of the same total length produced by a flint prism and by a crown prism are not of the same length from red to yellow, yellow to green, etc.,

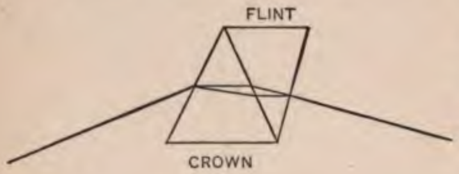

Fig. 104.

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* Strictly, of definite frequency, inasmuch as the wave-length is halved, for example, when a given wave-train passes into a medium in which its velocity is halved.

so that a flint prism cannot be made to completely neutralize the dispersion of a crown prism.

A compound lens, to be achromatic, must satisfy the condition expressed by equation (12), which is derived as follows: Let μ' and μ'' be the refractive indices of crown glass for red and blue respectively, and μ_1' and μ_1'' the corresponding refractive indices of flint glass. Let *C*, Fig. 105, be a converging crown-glass lens and *F* a diverging flint-glass lens. Let h be the thickness* of the crown lens at the center and h' the thickness of the flint lens at the edge, and let d be the diameter of the lenses. Consider the retardation of the central portion, relative to the edge portion, of a plane wave of red light. The retardation by the crown lens is $(\mu' - 1)h$, and the retardation by the flint lens is $-(\mu_1' - 1)h'$ (see Art. 54); the total retardation is therefore $[(\mu' - 1)h - (\mu_1' - 1)h']$. Similarly, the total retardation of the central portion of a plane wave of blue light is $(\mu'' - 1)h - (\mu_1'' - 1)h'$. If these two waves are to be focussed at the same point, the two retardations must be equal, which gives at once:

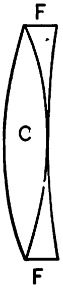

Fig. 105.

$$\frac{h'}{h} = \frac{\mu' - \mu''}{\mu_1' - \mu_1''} \tag{12}$$

This is the necessary relation between h and h' to give achromatism. The absolute values given to h and h' depend upon the diameter of the lens and the desired focal length. The curvatures of the various surfaces, in so far as they are not fixed by the absolute values of h and h' may be chosen so as to make the system aplanatic.

Remark. — A prism of crown glass which gives the same mean deflection as a prism of flint glass gives a much broader spectrum than the latter, so that two such prisms may be arranged to give a very good spectrum, the middle part of which is not deflected. Such an arrangement of prisms is used in the *direct-vision spectroscope.* The action is rendered more satisfactory by using a number of prisms as shown in Fig. 106.

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* The crown lens is assumed to come to a sharp edge and the flint lens is assumed to be of zero thickness at the center for the sake of simplicity.

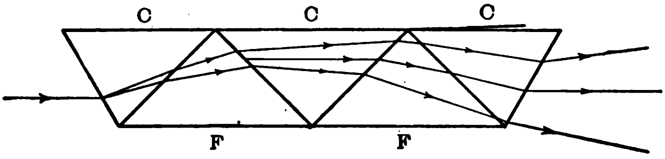

Fig. 106.

95. The spectroscope. — In the spectrum as obtained by Newton (explained in Art. 93) the beam of white light has some breadth and the various homogeneous beams, which are produced by the dispersion of the prism, are each as wide as the original beam of white light. Therefore the various homogeneous beams overlap each other greatly. This difficulty is overcome by means of the *spectroscope.* In this instrument, Fig. 107, the light to be analyzed is passed through a very narrow slit, *S*, between two straight metal edges. This slit, which may be regarded as the source of the light, is at the principal focus of an achromatic lens *L*, called the *collimating* lens. The spherical waves from the various points of the slit are plane after passing through *L*. These plane * waves pass through the prism *P*, after which each homogeneous component of the light (*i. e.*, each simple wave-train) appears to have come from a distinct slit, and the lens *L'* forms images of the slit side by side at *RV*, one for each homogeneous component. This band of images is called a *spectrum;* it is viewed by a magnifying glass or eye-piece *E*. The images of the slit are called the *lines* of the spectrum.

96. Continuous spectra. — When light from a hot solid or liquid is analyzed by a spectroscope a continuous band of images of the slit is produced at *RV*, Fig. 107. Such a spectrum is called a *continuous spectrum.* Light which gives a continuous spectrum contains wave-trains of every gradation of wave-length.

* Spherical waves are much distorted when refracted at a plane surface, as described in Art. 48; hence the importance of the collimating lens *L*.

The candle flame, the petroleum flame, and the gas flame give continuous spectra. The light from such flames is given off by hot particles of solid carbon. The lime light, the magnesium flame, the electric glow lamp, and the crater of the electric arc lamp give continuous spectra.

97. Bright-line spectra. — When the light from a hot vapor or gas is analyzed by a spectroscope a group of distinctly separated images of the slit is produced at *RV*, Fig. 107. Such a spectrum is called a *bright-line* spectrum inasmuch as the separate images of the slit appear as bright lines. Light which gives a bright-line spectrum contains only wave-trains of certain definite wave-lengths.

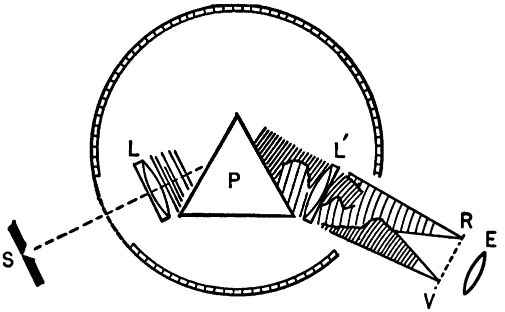

Fig. 107.

Every gas or vapor has a characteristic spectrum; that is, a characteristic grouping of images of the slit of a spectroscope. With increasing temperature, however, the spectrum of a given vapor grows more and more complex by the appearance of new lines which at the lower temperature were perhaps too dim to be seen. Thus the metals lithium, sodium, thallium and indium when vaporized in the flame of the Bunsen burner give each a single line. Lithium gives a brilliant red line, sodium a brilliant yellow line, thallium a brilliant green line, and indium a blue line. The sodium line is really two images of nearly equal brightness very near together. These two images appear as one unless the slit is very narrow, the dispersion large, and the definition of the spectroscope good. When the metals are vaporized in the electric arc each gives a number of images of the slit. Sodium, for example, gives some sixteen lines, and thallium some twenty-five lines in the visible spectrum ; and lines have been found in the ultra-violet region by photography and in the infra-red region by means of the bolometer.

The most complex bright-line spectrum is probably the spectrum given by iron vapor. Thousands of lines have been identified in the spectrum of this vapor and listed, and the catalogue is doubtless incomplete.

Remark 1. — Many stars and nebulæ give bright-line spectra from which we infer that these stars and nebulæ consist of hot masses of vapor.

Remark 2. — The spectroscope furnishes a valuable means to the chemist for detecting the presence of the various metals in an unknown mixture. The mixture is vaporized, usually in the flame of a Bunsen burner, the light from the vapor is analyzed by the spectroscope, and the presence, in the spectrum, of the characteristic group of lines corresponding to any element is indicative of the existence of that element in the mixture.

98. Dark-line spectra. — When an intense beam of white light, containing all wave-lengths, is passed through a relatively cool vapor or gas and then analyzed by the spectroscope, missing images of the slit, or *dark lines*, are seen where bright lines would be in the direct spectrum of the vapor or gas. That is, a relatively cool vapor absorbs those particular wave-trains which the vapor itself gives off when hot.

This relation between the bright-line spectrum of a vapor or gas and the dark-line spectrum of the vapor or gas, obtained as above described, was discovered by Bunsen and Kirchhoff in 1865. Using the flame of a Bunsen burner supplied with sodium vapor by the vaporization of common salt, they obtained the ordinary bright-line spectrum of sodium. Then, passing an intense beam of light from a lime light through the Bunsen flame into the slit, it was found that the absorption of the sodium vapor was such as to leave relatively dark lines in place of the bright lines given by the flame alone. This experiment has become classical under the name of *the reversal of the sodium lines.*

The most striking dark-line spectrum is the *solar spectrum,* which shows a great number of dark lines. These dark lines in the solar spectrum were first studied by Fraunhofer in 1819 and they are called *Fraunhofer's lines.* The more prominent of these lines are designated by the letters A, B, C, D, E, b, F, G, H_1, and H_2 in order, beginning with the red end of the spectrum. Fig. 108 shows the visible portion of the solar spectrum as it appears in a small spectroscope.

Groups of dark lines in the solar spectrum are found to coincide with the groups of bright lines given by iron vapor, sodium vapor, hydrogen and other elements, which is an indication that relatively cool vapors of these substances exist in the sun's atmosphere. Certain dark lines in the solar spectrum vary in intensity with the altitude of the sun above the horizon. These lines are due to the absorption of the earth's atmosphere.

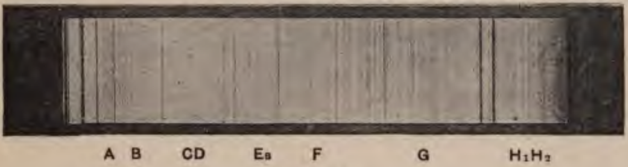

Fig. 108.

Many stars give dark-line spectra, more or less like the solar spectrum, from which we infer that their chemical and physical constitution is like that of our sun, which is a comparatively dense and extremely hot radiating body surrounded by relatively cool vapors.

99. The spectrometer. — This is a spectroscope which is provided with a divided circle by means of which the lines of the spectrum may be definitely located. The essential parts of a spectrometer are shown in Fig. 107. The lenses L' and E are respectively the objective and eye-piece of a telescope, which turns about a pivot at the center of the circle and is provided with a cross wire in the focal plane RV.

Consider a beam of homogeneous light which, upon emergence from the prism, is focussed by the lens L'. When the telescope is set so that its cross wire coincides with the line of the spectrum, which is due to the homogeneous beam in question, then the axis of the telescope is parallel to the beam, and the reading upon the divided circle, of a vernier which moves with the telescope, determines the location of the spectral line.

Determination of refractive index. — An important use of the spectrometer is in the determination of refractive indices. The glass to be studied is made into a prism of which the angle ϕ, Fig. 38, is measured, and the deflection (minimum) α, Fig. 38, is determined by means of the spectrometer; whence the refractive index of the glass, corresponding to the wave-length of light used, may be calculated as explained in Art. 46.

100. The spectrophotometer.—This instrument is a spectroscope arranged for determining the *composition* of light; that is, for determining the relative intensities of the various homogeneous components or wave-lengths of a given light. The measurements consist in comparing the intensity of each part of the spectrum of a given source of light with the intensity at the same part of the spectrum of a constant light source which has been selected as a standard. A convenient form of spectrophotometer is described in the chapter on photometry.

CHAPTER VIII.

INTERFERENCE AND DIFFRACTION.

101. Interference of waves from similar sources. — Consider two sources, O and O', Fig. 109*a*, which send out waves in exactly the same rhythm. Two such sources are called *similar sources.* The sources O and O' may be identically similar disturbances on the surface of water, or they may be two tuning forks in unison sending out sound waves, or O may be a luminous point and O' its image in a mirror. We shall limit the immediate discussion to the case in which O and O' send out simple wave-trains. Let λ be the wave-length of these wave-trains. Consider a point q whose distance from O is equal to its distance from O', or differs from it by a *whole number of wave-lengths,* $n\lambda$. The wave-trains from O and O' are continually alike in phase at such a point, and they work together to produce disturbance there. On the other hand, at a point p, whose distance from O differs

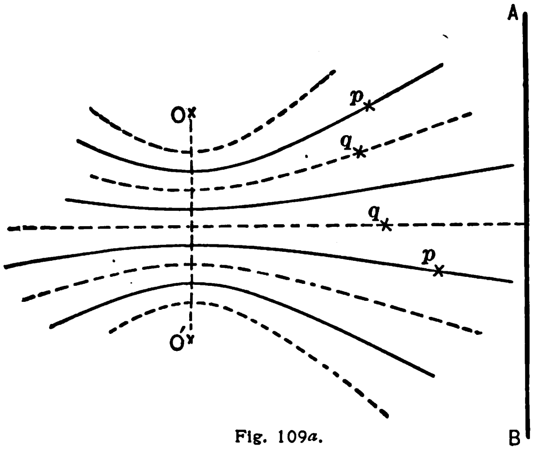

Fig. 109*a*.

from its distance from O' by an odd number of half wave-lengths, the wave-trains are continually opposite in phase and tend to annul each other.

All the points q which satisfy the above condition lie on the dotted hyperbolas,* Fig. 109*a*, of which O and O' are the foci. These dotted hyperbolas intersect the line OO' at equal intervals $(\frac{1}{2}\lambda)$. One is a straight line bisecting OO'.

All the points p which satisfy the above condition lie on the full-line hyperbolas midway between the dotted hyperbolas.

Along the dotted lines of Fig. 109*a* the disturbance due to the two sources O and O' is great, and along the full lines the disturbance is small or in some cases zero. If the light from O and O' falls on a screen AB, bright bands of illumination will be produced where AB intersects the dotted hyperbolas (hyperboloids of revolution about OO' as an axis), and dark bands will be left along the intersections of AB with the full-line hyperboloids.

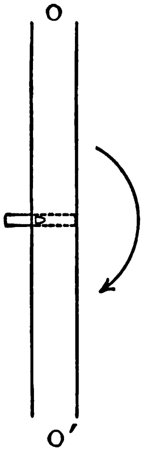

Fig. 109*b*.

This phenomenon is called *interference*. The light and dark bands on the screen are called *interference fringes*.

Example. — OO', Fig. 109*b*, is a large brass tube, about a meter in length, at the center of which a small, shrill-sounding whistle is inserted. When this whistle is blown the ends O and O' become similar sources of sound waves, and if there are no large objects near, to introduce complications by reflection of the waves, the state of affairs represented in Fig. 109*a* will be produced, namely, a series of hyperboloidal sheets *qqq* where the resultant sound is loud, and a series of intervening hyperboloidal sheets *ppp* where the resultant sound is weak. If the person who blows the whistle turns slowly round, rotating the tube OO' as indicated by the curved arrow, then the entire system of hyperboloidal sheets also rotates and a second person standing at some distance hears pulsations of loudness as the *p* and *q* regions pass by him.

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* The hyperbola is the locus of a point whose distances from two fixed points have a constant difference.

102. Distribution of fringes over a screen.—Consider two similar sources *O* and *O′*, Fig. 110, which illuminate a screen *AB*. It is required to find the distances from *C* of the various bright and dark bands on the screen. Let λ be the wave-length of the light given out by *O* and *O′*, let *a* be the distance between the similar sources *O* and *O′*, let *b* be the distance *OC*, and *assume that a is very small compared with b.*

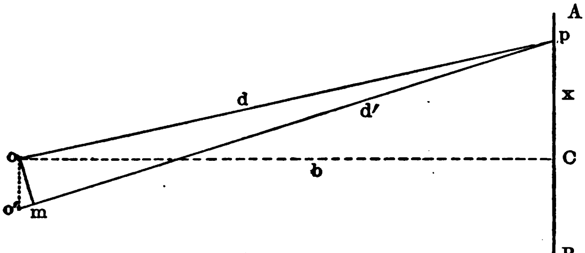

Fig. 110.

Let *d* and *d′* be the respective distances of *O* and *O′* from a point *p* on the screen, and let *x* be the distance *pC*. When the difference $(d' - d)$ is equal to $n\lambda/2$, then the point *p* is at the middle of a bright band if *n* is an even integer, and at the middle of a dark band if *n* is an odd integer.

Draw the line *Om* so that *mp* equals *Op*, then since *OC* is very large compared with *OO′*, the triangles *OmO′* and *OpC* are sensibly similar so that:

$$\frac{O'm}{OO'} = \frac{x}{b},$$

or

$$\frac{d' - d}{OO'} = \frac{x}{b}.$$

Writing $n\lambda/2$ for $(d' - d)$, and *a* for *OO′*, and solving for *x* we have

$$x = \frac{nb\lambda}{2a} \tag{13}$$

which expresses the distances x from the center of the screen to the various fringes. The distances to the dark fringes are found by putting for n the successive odd numbers 1, 3, 5, etc., and the distances to the bright fringes are found by putting for n the successive even numbers 2, 4, 6, etc.

For light, λ is always very small and the ratio b/a must be very large if the successive fringes are to be far enough apart to be distinguishable. Equation (13) enables the calculation of λ when x, a and b, have been observed. The number n may be easily counted, being two for the first bright band, four for the second and so on. This is, perhaps, the simplest way to measure the wave-length of light, and although a method of vastly greater accuracy will be explained in connection with the diffraction grating it may be well to call attention here to the fact that for the extreme violet light of the spectrum, the value of λ is about $39 \cdot 10^{-6}$ centimeters, and for the extreme red light about $75 \cdot 10^{-6}$ centimeters.

103. Colored fringes.—If the similar sources OO', Figs. 109*a* and 110, give off white light instead of homogeneous light, then a separate set of fringes will be produced on the screen by each homogeneous component of the white light, that is, by each wave-length of light, and the effect on the screen will be the superposition of all the fringes thus produced. The central fringe will be white, for this is a bright fringe for every wave-length. Passing out from this bright center we come first to the region where violet light, having the shortest wave-length is extinguished, leaving the illumination of the complementary* hue; and so on for blue, green, yellow and red light in order.

104. Arrangements for producing interference fringes.

(*a*) *Newton's arrangement.*—Newton, who was among the first to observe interference phenomena, employed two narrow slits, close together, as described in Art. 113.

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* Two colors which when mixed give white light are said to be *complementary* to each other. See Chapter X.

(*b*) *Fresnel's mirrors.*—Light from a very small source *S*, Fig. 111, falls upon two mirrors *M*, *M'*, as shown. After reflection from *M* and *M'* the direction of the rays is such that the light seems to come from the points *O* and *O'*, and interference fringes are produced on the portion *a* of the screen where the reflected rays from *M* and *M'* overlap.

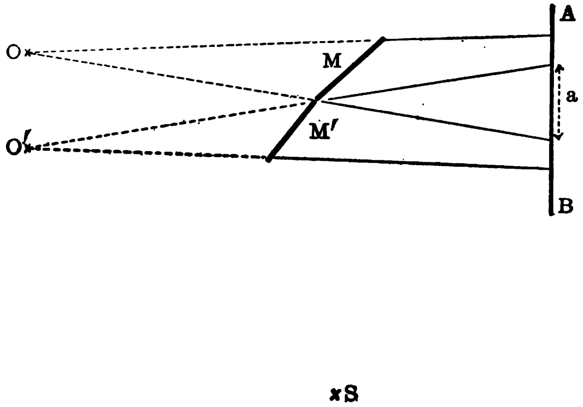

Fig. 111.

(*c*) *Lloyd's mirror.*—The portion *a* of a screen *AB*, Fig. 112, receives light directly from a small source *O* and also indirectly by reflection from a mirror as shown. This portion of the screen is therefore in effect illuminated by the two similar sources *O* and *O'* and interference fringes are produced.

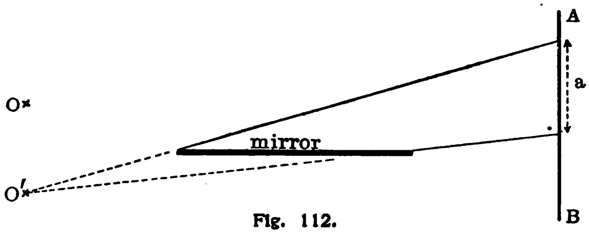

Fig. 112.

Sound fringes.—Using a shrill whistle as a source of sound waves of about 2½ centimeters wave-length, Stevens and Mayer have obtained marked interference effects by using arrangements similar to Fresnel's mirrors and to Lloyd's mirror. The regions of intense disturbance were indicated by a sensitive flame. Such a flame is depicted in Fig. 113. When undisturbed, the flame is long and narrow (*b*); when disturbed it changes to the shorter form (*a*).

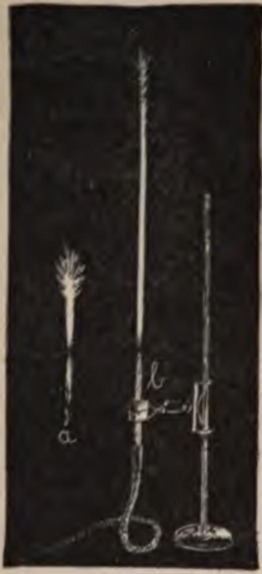

Fig. 113.

105. Beats. — If the source *O*, Fig. 109*a*, has a slightly greater frequency than the source *O′*, giving out waves of slightly shorter wave-length, then the hyperboloidal sheets *ppp* . . . *qqq* . . . will recede slowly from *O* and slowly approach *O*, a given sheet moving from a *p*-position to a *q*-position while *O* gains half a vibration over *O′*. Thus if *O* and *O′* are tuning forks having slightly different frequencies, the regions of loud and weak sound will move as stated and a stationary ear will hear pulsations of loudness as these regions (hyperboloidal sheets) sweep by. These pulsations of loudness are called *beats*. There is no observable phenomenon in light corresponding to beats in sound inasmuch as *approximate* equality of frequency of two light sources cannot be realized.

106. The colors of thin plates. — Thin plates present very striking interference phenomena, which are of common occurrence. The colors of soap films, and of films of oil on water, are familiar examples. The action of a thin film in producing interference is briefly as follows:

Consider a simple train of waves *T*, Fig. 114, of wave-length λ, incident upon a thin transparent plate *PP*. A portion T' of this train is reflected from the surface *A* (with change of phase, see Art. 22). The remainder of the train, passing on through the plate, reaches the second surface *B*, and is partly reflected (without change of phase). This portion, after passing back through the plate, emerges (in part) into the air and travels as the train T'' parallel to and partly overlapping the train T'. It is in the overlapping portions of T' and T'' that interference is produced.

The distance $2a$,* Fig. 114, is the amount that wave-train T'' has fallen behind wave-train T', and since T' has been reflected with change of phase, and T'' has been reflected without change of phase, it is convenient to think of $(2a + \lambda / 2)$ as the total retardation of one wave-train with respect to the other. Now if this total retardation is a whole number of wave-lengths (*i. e.*, if $2a$ is an odd number of half wave-lengths) the crests of wave-train T' will coincide with the crests of wave-train T'' and an intense beam of light will be the result. If, however, the total retardation is an odd number of half wave-lengths (*i. e.*, if $2a$ is a whole number of wave-lengths) the crests of wave-train T' will coincide with the hollows of wave-train T'' and these wave-trains will tend to annul each other.

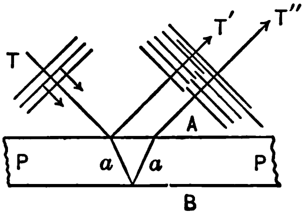

Fig. 114.

If the incident light T is non-homogeneous (white light), then all those homogeneous components of the white light are greatly strengthened whose half wave-lengths are contained in the distance $2a$ an odd number of times, and those homogeneous components are greatly weakened whose half wave-lengths are contained in the distance $2a$ an even number of times.

If the plate is very thin, *i. e.*, if a is small, then, of the various visible homogeneous components of white light ($\lambda = 75 \times 10^{-6}$ centimeters for red, to $\lambda = 39 \times 10^{-6}$ centimeters for violet) only one or two will satisfy the above conditions, and the plate will appear brilliantly colored. If the plate is thick, however, then a great number of the components will satisfy the condition. In this case the strengthened components and the weakened components will be distributed more or less evenly throughout the spectrum, and the plate will not show perceptible color.

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* This distance $2a$ is taken as the equivalent distance in air. The actual distance in the plate must be multiplied by the index of refraction of the plate to give its equivalent value of air.

In practice the mirrors *C* and *D* are not exactly at right angles so that the field of view of the telescope is crossed by bright and dark bands. These bands move sidewise across the field of view when the mirror *C* is moved in the direction of *a* or *b* and the movement of the mirror is equal to $n\lambda/2$, where n is the number of dark bands which pass by a given point of the field during the movement.

DIFFRACTION.

109. Diffraction.—The spreading of a wave disturbance into the region behind an obstacle is called *diffraction.* We shall first consider this phenomenon to the extent of determining the order of magnitude of an obstacle in order that it may screen off a disturbance from a point behind it; this will give an idea of the degree of approximation to which a wave disturbance may be thought of as propagated in straight lines. We shall then consider what takes place in a region throughout which a wave disturbance does spread by diffraction.

110. Half-period zones with reference to a point.—Consider a simple train of plane waves *TT*, Fig. 116, of very short wave-length λ, approaching the point *O* as indicated by the arrows; the wave fronts being parallel to a reference plane *AB* distant *b* from *O*.

At the instant that a wave of the train TT reaches the plane AB, the whole plane AB is a sheet of disturbance; and we are to imagine, according to Huygens' principle, that at this instant wavelets start out from every point of AB, the disturbance at O being due to the combined action of these wavelets. Now the various points of the plane *AB* are at unequal distances from *O* and since wavelets start from every part of *AB* simultaneously it is evident that some wavelets when they reach *O* will be opposite in phase to others and some will be in phase with others. It is necessary to distinguish (*a*) those portions of the plane *AB* the wavelets from which work together at *O* and (*b*) those portions the wavelets from which work oppositely to the wavelets from the portions (*a*).

Draw about O as a center a sphere of radius b, another sphere of radius $b + \lambda/2$, another of radius $b + 2\lambda/2$, another of radius $b + 3\lambda/2$, and so on. These spheres will cut the plane AB into circular zones as shown in Fig. 117. Let these zones be numbered as indicated in Fig. 117. The outside radius r_n of the nth zone may be found from the triangle, Fig. 118, from which we have

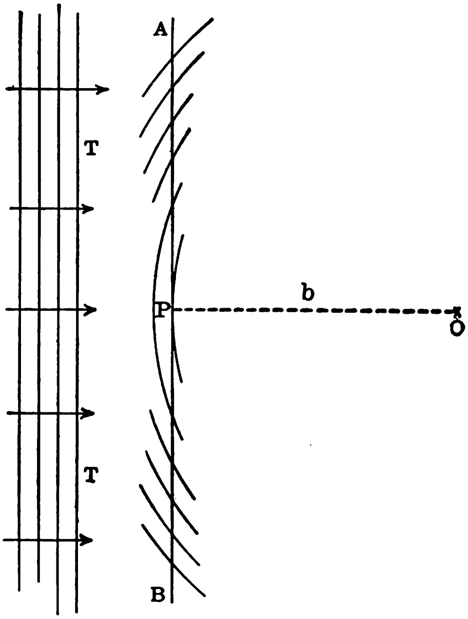

Fig. 116.

$$r_n^2 + b^2 = \left(b + \frac{n\lambda}{2}\right)^2;$$

whence, since $(n\lambda/2)^2$ is negligible, we have

$$r_n = \sqrt{nb\lambda} \qquad (14)$$

The disturbance reaching O from the nth zone is on the whole opposite in phase to the disturbance reaching O from the $(n + 1)$th zone, since the $(n + 1)$th zone is on the whole half a wave-length

In practice the mirrors C and D are not exactly at right angles so that the field of view of the telescope is crossed by bright and dark bands. These bands move sidewise across the field of view when the mirror C is moved in the direction of a or b and the movement of the mirror is equal to $n\lambda/2$, where n is the number of dark bands which pass by a given point of the field during the movement.

DIFFRACTION.

109. Diffraction. — The spreading of a wave disturbance into the region behind an obstacle is called *diffraction*. We shall first consider this phenomenon to the extent of determining the order of magnitude of an obstacle in order that it may screen off a disturbance from a point behind it; this will give an idea of the degree of approximation to which a wave disturbance may be thought of as propagated in straight lines. We shall then consider what takes place in a region throughout which a wave disturbance does spread by diffraction.

110. Half-period zones with reference to a point. — Consider a simple train of plane waves TT, Fig. 116, of very short wavelength λ, approaching the point O as indicated by the arrows; the wave fronts being parallel to a reference plane AB distant b from O.

At the instant that a wave of the train TT reaches the plane AB, the whole plane AB is a sheet of disturbance; and we are to imagine, according to Huygens' principle, that at this instant wavelets start out from every point of AB, the disturbance at O being due to the combined action of these wavelets. Now the various points of the plane AB are at unequal distances from O and since wavelets start from every part of AB simultaneously it is evident that some wavelets when they reach O will be opposite in phase to others and some will be in phase with others. It is necessary to distinguish (a) those portions of the plane AB the wavelets from which work together at O and (b) those portions the wavelets from which work oppositely to the wavelets from the portions (a).

Draw about O as a center a sphere of radius b, another sphere of radius $b + \lambda/2$, another of radius $b + 2\lambda/2$, another of radius $b + 3\lambda/2$, and so on. These spheres will cut the plane AB into circular zones as shown in Fig. 117. Let these zones be numbered as indicated in Fig. 117. The outside radius r_n of the nth zone may be found from the triangle, Fig. 118, from which we have

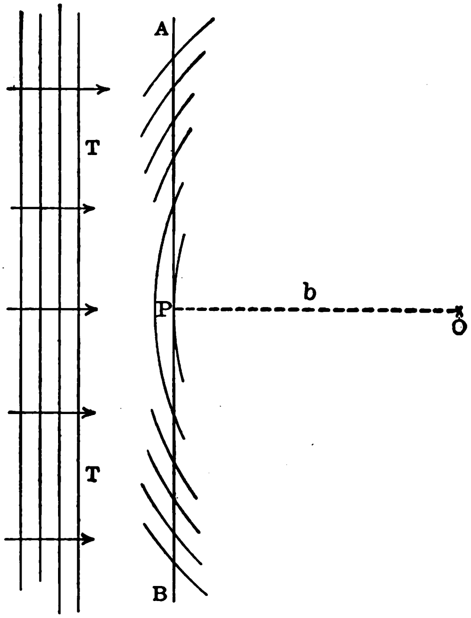

FIG. 116.

$$r_n^2 + b^2 = \left(b + \frac{n\lambda}{2}\right)^2;$$

whence, since $(n\lambda/2)^2$ is negligible, we have

$$r_n = \sqrt{nb\lambda} \tag{14}$$

The disturbance reaching O from the nth zone is on the whole opposite in phase to the disturbance reaching O from the $(n + 1)$th zone, since the $(n + 1)$th zone is on the whole half a wave-length farther from *O*. Therefore all the odd-numbered zones work together to produce disturbance at *O*, and all the even-numbered zones work together to oppose the effect of the odd-numbered zones.*

The intensity of the disturbance reaching *O* from a zone is less and less the higher the order of the zone; but two adjacent zones of high order produce at *O* disturbances that are almost exactly equal in intensity and opposite in phase and their effect therefore is to annul each other.* Therefore *the actual disturbance at O comes from zones of low order.*

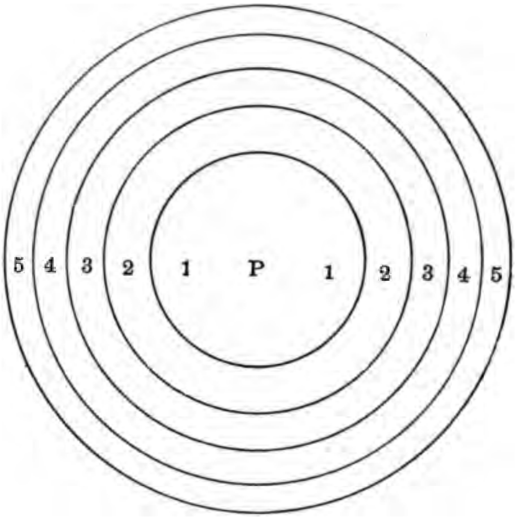

Fig. 117.

Examples. — (*a*) The wave-length of yellow light is about $6 \cdot 10^{-5}$ centimeters. If *b*, Fig. 116, is 10 centimeters, the diameter of, say, the tenth half-period zone is about 0.16 centimeter; which is the order of magnitude of the obstacle which, placed at *P*, Fig. 116, would effectually screen the point *O* from yellow light.

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* These statements are sufficiently accurate and definite for present purposes. For a more complete treatment of this subject see Preston's "Light," Chapters III. and IX.

(*b*) The wave-length of a medium pitch sound is about 50 centimeters. If *b*, Fig. 116, is 40,000 centimeters, the diameter of the tenth half-period zone is about 8,800 centimeters; which is the order of magnitude of the obstacle which placed at *P*, Fig. 116, would effectually screen the point *O* from a medium pitch sound.

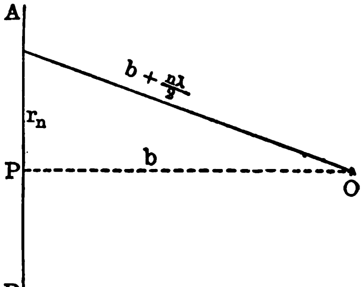

Fig. 118.

The conception of the ray.—In case of light with its extremely short wave-lengths the disturbance at *O*, Fig. 116, comes from a small portion of the plane *AB* in the neighborhood of the point *P*, and this disturbance may therefore be thought of as traveling along the line *PO*. This line *PO* is called a *ray*.

The zone-plate.—A transparent plate having opaque bands painted upon it so that when placed in the position of *AB*, Fig. 116, the even-numbered or odd-numbered zones will be obscured is called a zone-plate. A train of waves *TT*, Fig. 116, passing through such a plate produces very intense illumination at the point *O*.

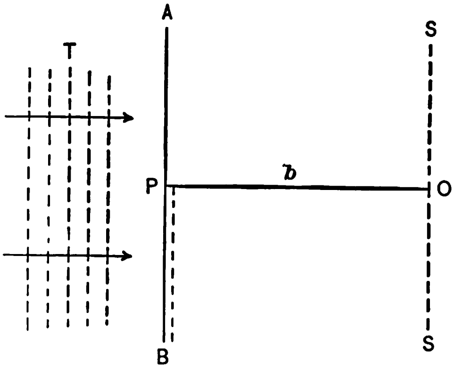

Fig. 119.

111. Half-period zones with reference to a line.—Let *TT*, Fig. 119, be a train of plane waves of wave-length λ approaching a screen *SS*. Let *O* be a line perpendicular to the paper along which the illumination produced by *TT* is to be considered, with a view to the determination of the effect of an obstacle in the fixed plane *AB*, distant *b* from *O*. With *O* as an axis, describe circular cylinders (not shown in the diagram), one of which has a radius b, the next a radius $b + \lambda/2$, the next a radius $b + \lambda$, the next a radius $b + 3\lambda/2$, and so on. These cylinders will cut the plane AB into bands or zones parallel to O. Calling the middle zone No. 1, the distance from P to the outer edge of the nth zone is $\pm\sqrt{nb\lambda}$, as explained in Art. 110. The middle zone is the broadest; the successive zones grow narrower above and below P and approach the limiting width $\lambda/2$.

For the purpose of the following discussion it is sufficient to know that *the portion of the disturbance along O which is contributed by the nth zone is, on the whole, opposite in phase to that contributed by the* $(n + 1)$*th zone, and that this contribution grows less and less as n increases.*

The illumination along O produced by the unobstructed train of waves is called *normal* illumination.

112. Diffraction past the straight edge of a large obstacle.

(*a*) *For points outside of the geometrical shadow.* — When the edge of the obstacle (which is parallel to O) is at P, as shown by the dotted line in Fig. 119, half of the middle zone is cut off, together with all the zones below P, and the illumination along O is one quarter* normal. If the obstacle is moved downwards, the middle zone will gradually be uncovered and the illumination along O will increase, reaching a maximum considerably greater than normal at about the time when the middle zone is wholly† uncovered. As the next zone (No. 2) becomes uncov-

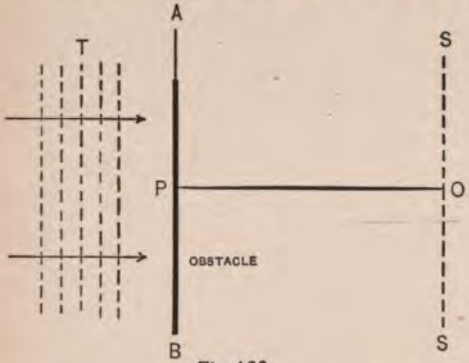

Fig. 120.

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* The *amplitude* of the disturbance at O is *one half* normal, and therefore (Art. 19) the intensity is one quarter normal.

† Strictly, when 0.87 of the lower half of the middle zone is uncovered.

ered, the illumination at *O* falls off, because the contribution from this zone is nearly opposite in phase to the contribution from the middle zone, and reaches a minimum considerably less than normal at about the time when the second zone is wholly uncovered. As the third zone becomes uncovered, the illumination along *O* reaches a maximum, and so on. The maxima and minima grow less and less pronounced as the obstacle moves down, and the illumination along *O* becomes sensibly constant and equal to normal when the obstacle has receded so far as to uncover as many as six or eight zones below *P*.

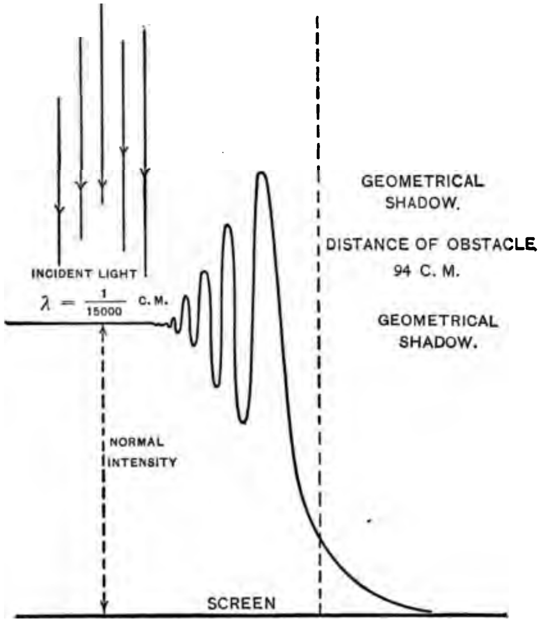

Fig. 121.

(*b*) *For points within the geometrical shadow.*—Let the unobstructed portion of *AB*, Fig. 120, be broken up into half-period zones *beginning at the edge of the obstacle.* In this case the band next the edge is more effective in producing disturbance at *O* than the next band, the effect of which at *O* is opposite in phase and so on. There is, therefore, a slight resultant disturbance, or illumination, at *O* produced by all the bands above the edge of the obstacle. This resultant illumination has a quarter of its normal value when the edge is opposite *O*. As the edge of the obstacle moves upwards from *P* towards *A*, the illumination at *O* falls off steadily.

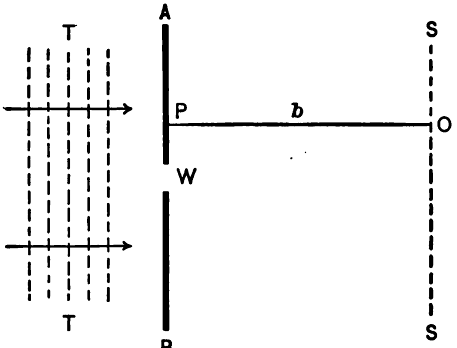

Fig. 122.

The variation of illumination at *O*, for various positions of the edge of the obstacle, as explained above, corresponds exactly to the various degrees of illumination *at different points on the screen* for a *fixed position of the obstacle*. The ordinates of the curve, Fig. 121, represent the degrees of illumination at various points on the screen as described above under (*a*) and (*b*), but for a fixed position of the obstacle.

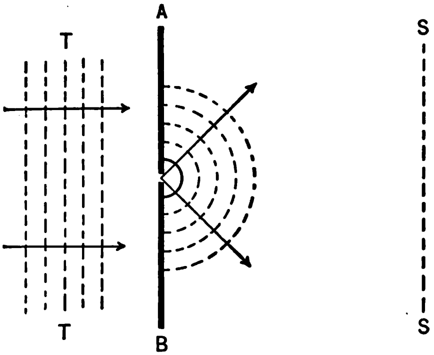

Fig. 123.

113. Diffraction through a slit. — Consider the illumination reaching the point *O*, Fig. 122, through the slit *W*. There are three cases as follows:

(*a*) *When the slit is less than half a wave-length in breadth* there can never be more than one half-period zone in the slit and the light from the slit spreads out with approximate uniformity as indicated in Fig. 123. Two such slits near together act as *similar sources* * and produce interference fringes on the screen *SS*, as explained in Art. 101. The regions of maximum and minimum disturbance, the p and q regions described in Art. 101, produced by the sound coming from a shrill whistle through two adjacent narrow slits in a wall, have been traced by Stevens and Mayer, and by Rayleigh, using the sensitive flame as an indicator as explained in Art. 104.

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* Provided the light *TT* has come from a very small source.

(*b*) *When the slit is of medium width, say 1,000 wave-lengths.* In this case the illumination at O, Fig. 122, is of course zero when the slit W is far below P. As the slit moves up along AB, the slit contains fewer and fewer half-period zones inasmuch as the zones become broader. When there is an even number of zones in the slit they neutralize each other's action at O and the illumination at O is zero. When there is an odd number of zones in the slit, the effect at O of one of them is left outstanding and the illumination at O is a maximum. The intensities of these maxima increase as the slit approaches P and fall off again as the slit moves on past P towards A.

The variation of illumination at O for various positions of the slit, as explained above, corresponds exactly to the various degrees of illumination *at different points on the screen* for a *fixed position of the slit.*

(*c*) *When the slit is very broad, many thousands of wave-lengths.* Each edge of the slit acts like the edge of a single obstacle as described in Art. 112.

114. Diffraction past the edges of a narrow strip.—Consider an obstacle OO', Fig. 124. The illumination on the screen outside the geometrical shadow varies according to a law which is nearly the same as that which applies when the obstacle is infinitely broad. Inside the geometrical shadow ab, the light comes mostly from regions very near the edges O and O', so that O and O' act as similar sources and produce interference fringes similar to those described in Art. 101.

One of the few cases in which the phenomena of the diffraction of light occur to ordinary observation is that of the shadows of wires and twigs cast upon frosted windows by a distant arc light.

In practice the mirrors C and D are not exactly at right angles so that the field of view of the telescope is crossed by bright and dark bands. These bands move sidewise across the field of view when the mirror C is moved in the direction of a or b and the movement of the mirror is equal to $n\lambda/2$, where n is the number of dark bands which pass by a given point of the field during the movement.

DIFFRACTION.

109. Diffraction.—The spreading of a wave disturbance into the region behind an obstacle is called *diffraction*. We shall first consider this phenomenon to the extent of determining the order of magnitude of an obstacle in order that it may screen off a disturbance from a point behind it; this will give an idea of the degree of approximation to which a wave disturbance may be thought of as propagated in straight lines. We shall then consider what takes place in a region throughout which a wave disturbance does spread by diffraction.

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At the instant that a wave of the train TT reaches the plane AB, the whole plane AB is a sheet of disturbance; and we are to imagine, according to Huygens' principle, that at this instant wavelets start out from every point of AB, the disturbance at O being due to the combined action of these wavelets. Now the various points of the plane AB are at unequal distances from O and since wavelets start from every part of AB simultaneously it is evident that some wavelets when they reach O will be opposite in phase to others and some will be in phase with others. It is necessary to distinguish (a) those portions of the plane AB the wavelets from which work together at O and (b) those portions the wavelets from which work oppositely to the wavelets from the portions (a).

Draw about O as a center a sphere of radius b, another sphere of radius $b + \lambda/2$, another of radius $b + 2\lambda/2$, another of radius $b + 3\lambda/2$, and so on. These spheres will cut the plane AB into circular zones as shown in Fig. 117. Let these zones be numbered as indicated in Fig. 117. The outside radius r_n of the nth zone may be found from the triangle, Fig. 118, from which we have

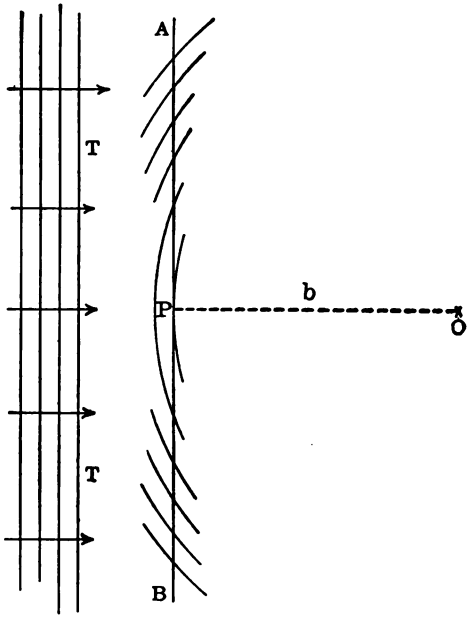

Fig. 116.

$$r_n^2 + b^2 = \left(b + \frac{n\lambda}{2}\right)^2;$$

whence, since $(n\lambda/2)^2$ is negligible, we have

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The disturbance reaching O from the nth zone is o
opposite in phase to the disturbance reaching O from the
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The intensity of the disturbance reaching *O* from a zone is less and less the higher the order of the zone; but two adjacent zones of high order produce at *O* disturbances that are almost exactly equal in intensity and opposite in phase and their effect therefore is to annul each other.* Therefore *the actual disturbance at O comes from zones of low order.*

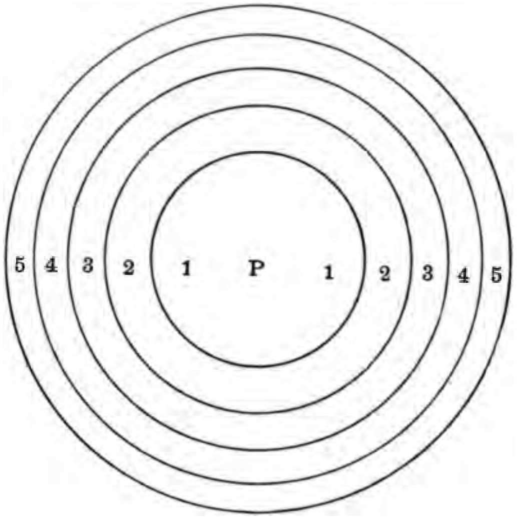

Fig. 117.

Examples. — (*a*) The wave-length of yellow light is about $6 \cdot 10^{-5}$ centimeters. If *b*, Fig. 116, is 10 centimeters, the diameter of, say, the tenth half-period zone is about 0.16 centimeter; which is the order of magnitude of the obstacle which, placed at *P*, Fig. 116, would effectually screen the point *O* from yellow light.

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* These statements are sufficiently accurate and definite for present purposes. For subject see Preston's "Light," Chapters III. and IX.

(*b*) The wave-length of a medium pitch sound is about 50 centimeters. If *b*, Fig. 116, is 40,000 centimeters, the diameter of the tenth half-period zone is about 8,800 centimeters; which is the order of magnitude of the obstacle which placed at *P*, Fig. 116, would effectually screen the point *O* from a medium pitch sound.

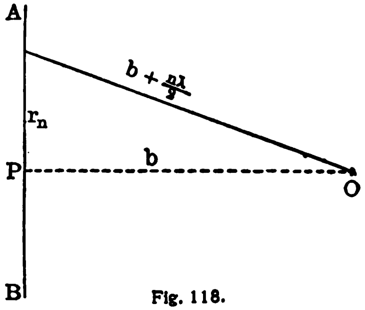

Fig. 118.

The conception of the ray.—In case of light with its extremely short wave-lengths the disturbance at *O*, Fig. 116, comes from a small portion of the plane *AB* in the neighborhood of the point *P*, and this disturbance may therefore be thought of as traveling along the line *PO*. This line *PO* is called a *ray*.

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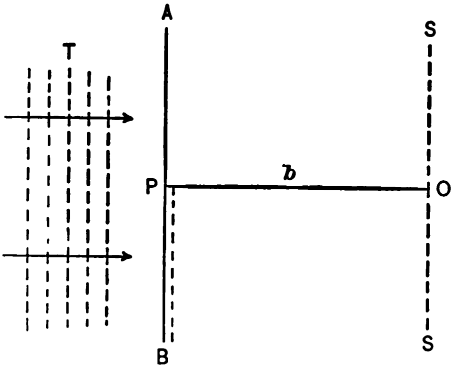

Fig. 119.

111. Half-period zones with reference to a line.— Let *TT*, Fig. 119, be a train of plane waves of wave-length λ approaching a screen *SS*. Let *O* be a line perpendicular to the paper along which the illumination produced by *TT* is to be considered, with a view to the determination of the effect of an obstacle in the fixed plane *AB*, distant *b* from *O*. With *O* as an axis, describe circular cylinders (not shown in the diagram), one of which has a radius b, the next a radius $b + \lambda/2$, the next a radius $b + \lambda$, the next a radius $b + 3\lambda/2$, and so on. These cylinders will cut the plane AB into bands or zones parallel to O. Calling the middle zone No. 1, the distance from P to the outer edge of the nth zone is $\pm \sqrt{nb\lambda}$, as explained in Art. 110. The middle zone is the broadest; the successive zones grow narrower above and below P and approach the limiting width $\lambda/2$.

For the purpose of the following discussion it is sufficient to know that *the portion of the disturbance along O which is contributed by the nth zone is, on the whole, opposite in phase to that contributed by the* $(n + 1)$*th zone, and that this contribution grows less and less as n increases.*

The illumination along O produced by the unobstructed train of waves is called *normal* illumination.

112. Diffraction past the straight edge of a large obstacle.

(*a*) *For points outside of the geometrical shadow.* — When the edge of the obstacle (which is parallel to O) is at P, as shown by the dotted line in Fig. 119, half of the middle zone is cut off, together with all the zones below P, and the illumination along O is one quarter[*] normal. If the obstacle is moved downwards, the middle zone will gradually be uncovered and the illumination along O will increase, reaching a maximum considerably greater than normal at about the time when the middle zone is wholly[†] uncovered. As the next zone (No. 2) becomes uncov-

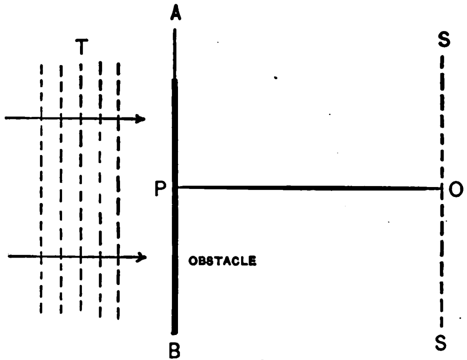

Fig. 120.

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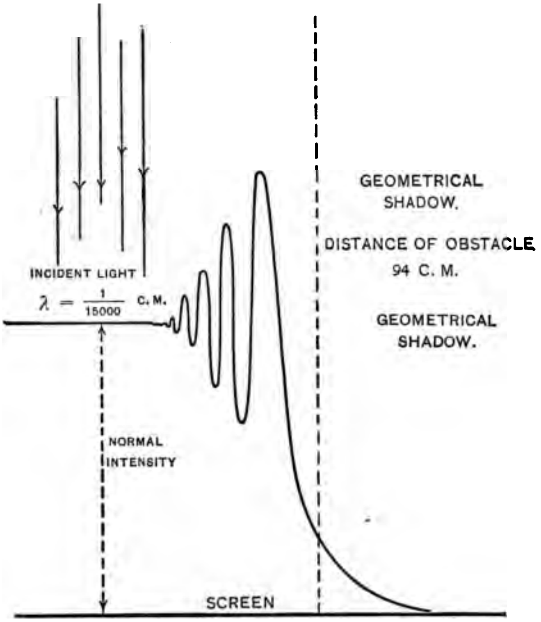
[*] The *amplitude* of the disturbance at O is *one half* normal, and therefore (Art.) the intensity is one quarter normal.

[†] Strictly, when 0.87 of the lower half of the middle zone is uncovered.

ered, the illumination at *O* falls off, because the contribution from this zone is nearly opposite in phase to the contribution from the middle zone, and reaches a minimum considerably less than normal at about the time when the second zone is wholly uncovered. As the third zone becomes uncovered, the illumination along *O* reaches a maximum, and so on. The maxima and minima grow less and less pronounced as the obstacle moves down, and the illumination along *O* becomes sensibly constant and equal to normal when the obstacle has receded so far as to uncover as many as six or eight zones below *P*.



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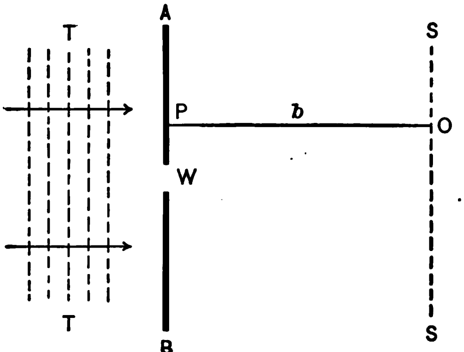
(*b*) *For points within the geometrical shadow.* — Let the unobstructed portion of *AB*, Fig. 120, be broken up into half-period zones *beginning at the edge of the obstacle.* In this case the band next the edge is more effective in producing disturbance at *O* than the next band, the effect of which at *O* is opposite in phase and so on. There is, therefore, a slight resultant disturbance, or illumination, at *O* produced by all the bands above the edge of the obstacle. This resultant illumination has a quarter of its normal value when the edge is opposite *O*. As the edge of the obstacle moves upwards from *P* towards *A*, the illumination at *O* falls off steadily.



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The variation of illumination at *O*, for various positions of the edge of the obstacle, as explained above, corresponds exactly to the various degrees of illumination *at different points on the screen* for a *fixed position of the obstacle.* The ordinates of the curve, Fig. 121, represent the degrees of illumination at various points on the screen as described above under (*a*) and (*b*), but for a fixed position of the obstacle.

113. Diffraction through a slit. — Consider the illumination reaching the point *O*, Fig. 122, through the slit *W*. There are three cases as follows:

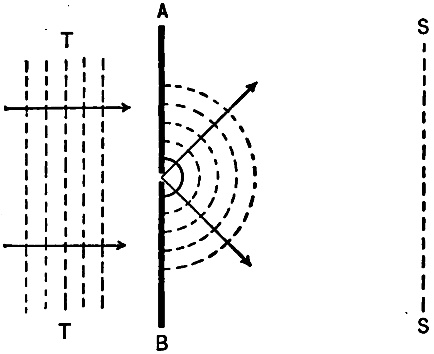

Fig. 123.

(*a*) *When the slit is less than half a wave-length in breadth* there can never be more than one half-period zone in the slit and the light from the slit spreads out with approximate uniformity as indicated in Fig. 123. Two such slits near together act as *similar sources* * and produce interference fringes on the screen *SS*, as explained in Art. 101. The regions of maximum and minimum disturbance, the *p* and *q* regions described in Art. 101, produced by the sound coming from a shrill whistle through two adjacent narrow slits in a wall, have been traced by Stevens and Mayer, and by Rayleigh, using the sensitive flame as an indicator as explained in Art. 104.

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* Provided the light *TT* has come from a very small source.

(*b*) *When the slit is of medium width, say 1,000 wave-lengths.* In this case the illumination at *O*, Fig. 122, is of course zero when the slit *W* is far below *P*. As the slit moves up along *AB*, the slit contains fewer and fewer half-period zones inasmuch as the zones become broader. When there is an even number of zones in the slit they neutralize each other's action at *O* and the illumination at *O* is zero. When there is an odd number of zones in the slit, the effect at *O* of one of them is left outstanding and the illumination at *O* is a maximum. The intensities of these maxima increase as the slit approaches *P* and fall off again as the slit moves on past *P* towards *A*.

The variation of illumination at *O* for various positions of the slit, as explained above, corresponds exactly to the various degrees of illumination *at different points on the screen* for a *fixed position of the slit.*

(*c*) *When the slit is very broad, many thousands of wave-lengths.* Each edge of the slit acts like the edge of a single obstacle as described in Art. 112.

114. Diffraction past the edges of a narrow strip.—Consider an obstacle *OO′*, Fig. 124. The illumination on the screen outside the geometrical shadow varies according to a law which is nearly the same as that which applies when the obstacle is infinitely broad. Inside the geometrical shadow *ab*, the light comes mostly from regions very near the edges *O* and *O′*, so that *O* and *O′* act as similar sources and produce interference fringes similar to those described in Art. 101.

One of the few cases in which the phenomena of the diffraction of light occur to ordinary observation is that of the shadows of wires and twigs cast upon frosted windows by a distant arc light.

The shadow is filled with fringes and bordered on both sides by fringes all more or less colored. When the wire or twig is large the fringes in the shadow are very close together. The bordering fringes do not vary perceptibly with width of wire or twig. These diffraction effects cannot be observed in shadows cast by an ordinary lamp or by the sun, inasmuch as the fringes are obliterated by the blurring effect which is always present in a shadow cast by a broad source of light. A point source of light, such as the arc light, is necessary.

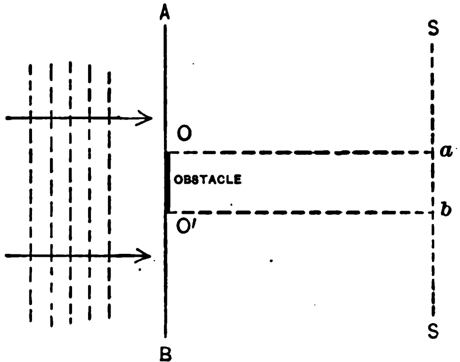

Fig. 124.

115a. The diffraction grating consists of a large number of equidistant parallel slits in an opaque screen. The action of the grating is most easily described in combination with a lens, for which purpose the following proposition will be useful: A plane wave *WW*, Fig. 125, approaching a lens *LL* obliquely as shown, is focussed at the point *F'* in the focal plane of the lens, the line *F'C* being perpendicular to the wave front. Since the wave *WW* is focussed at *F'*, every portion of *WW* must reach *F'* at the same instant, that is, *all rays, such as, R, R', etc., drawn from WW to F' have optically the same length.*

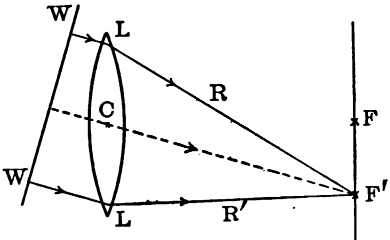

Fig. 125.

A simple train of plane waves *TT*, Fig. 126*a*, of wave-length λ approaches a diffraction grating *AB* as shown. Behind the grating is a lens *LL* of which the focal plane is represented by the line *F'''* *F*^{iv}. The portion of the wave-train *TT*, which passes through the grating, is focussed by the lens at a series of points F, F', F'', F''', F^{iv}, etc. Let d be the distance from center to center of adjacent slits. Draw the line GH from the center of the slit next below a, making the angle θ such that the distance from a to GH is $n\lambda$, the distance from b to GH is $2n\lambda$, the distance from c to GH is $3n\lambda$ and so on, n being any whole number. Then

$$d \sin \theta = n\lambda \qquad (15)$$

As each wave of the train TT strikes the grating, wavelets pass out simultaneously from all the slits and these wavelets are always in like phase along the line GH since this line is n full wave-lengths from slit a, $2n$ full wave-lengths from slit b, $3n$ full wave-lengths from slit c, and so on. Furthermore the point F' is at the same distance, optically, from every point in the line GH. Therefore, the wavelets from the various slits reach F' in like phase; the wavelets therefore work together to produce brilliant illumination at F'; F' is of course a line perpendicular to the plane of the paper and GH is a plane perpendicular to the plane of the paper.

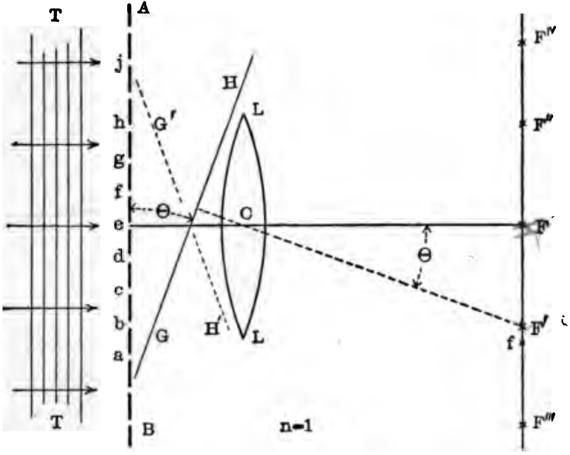

Fig. 126a.

The above argument applies unchanged to the line $G'H'$ and to the point F''. Furthermore, the lines GH and $G'H'$ and the points F' and F'' in the figure are drawn for the case $n = 1$. For $n = 2$ we have the two luminous points F''' and F^{iv}, for $n = 3$ we have an additional pair of luminous points still farther from the central point F and so on. The locations of the various points F', F'', F''', F^{iv}, etc., are determined by the values of the angle θ which satisfy equation (15.) Solving this equation for sin θ, and remembering what is stated above concerning the lines GH and $G'H'$, we have

$$\sin \theta = \pm \frac{n\lambda}{d} \qquad (16)$$

It is instructive to show why it is that, with a wave-train TT of given wave-length, the focal plane of the lens LL is illuminated only at the points F, F', F'', F''', etc., and nowhere else. Fig. 126*b* shows the line GH drawn in the position which satisfies equation (16) and it shows the wavelets from the various slits with their crests all touching GH at a given instant. Imagine the angle θ to be slightly increased, so that GH becomes tangent to wavelet w' from slit j, instead of wavelet w as shown. Corresponding to this new value of θ and new position of GH, we have the new point f, as shown in Fig. 126*a*, and the relative phases of the various wavelets when they reach f are the same as their relative phases when they reach the new position of GH. In fact a hollow of the wavelets from slit e is on the new line GH when the crest of the wavelets from slit j is on the new line GH, so that the wavelets from slit e annul the wavelets from slit j at the point f. Similarly, the wavelets from slit d annul the wavelets from slit i, the wavelets from slit c annul the wavelets from slit h and so on. Therefore the illumination at f is zero. In general, any point f for which the angle θ

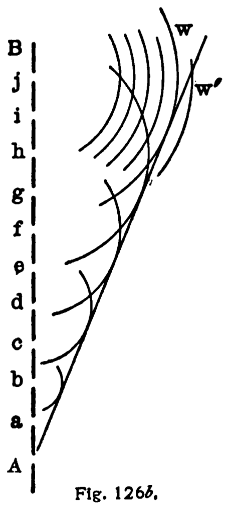

Fig. 126*b*.

does not satisfy equation (16), has either zero illumination or a negligible degree of illumination if the grating has a great many slits.

If the plane waves TT, Fig. 126*a*, are non-homogeneous (containing wave-trains of different wave-lengths), then there will be in the immediate neighborhood of the point F' (also in the immediate neighborhood of F'', F''', F^{iv}, etc.) a *group of luminous points* (lines) one for each wave-length that is contained in the incident light TT. This group of bright lines near F' is called a *diffraction spectrum.* The groups of bright lines near F' and F'' are called spectra of the first order ($n = 1$), the groups of bright lines near F''' and F^{iv} are called spectra of the second order ($n = 2$) and so on.

Example. — A striking illustration of the action of the diffraction grating as above described is furnished by looking through a finely and regularly woven umbrella web towards a distant arc light. The umbrella web is the grating, the lens of the eye takes the place of the lens LL, Fig. 126*a*, and the focal plane F''' F^{iv} is the retina. In this case each of the points F', F'', F''', etc., is a spectrum, and these spectra are duplicated above and below the plane of the paper in Fig. 126*a* on account of the crossing of two sets of slits in the web.

115b. The action of the grating upon obliquely incident light. — Consider the obliquely incident train of plane waves TT, Fig. 126*c*. The full line TT represents the position of a wave which has just reached the slit at B. In this case the wavelets do not pass out from all the grating slits simultaneously. The disturbance, however, which reaches the line GH from the line TT travels along the finely dotted lines the lengths of which are multiples of the lengths of the shortest ones a and b, so that if $a + b$ is a whole number of wave-lengths, then the wavelets from all the slits will reach GH in like phase and the point F''' will be brightly illuminated. Now $a = d \sin i$ and $b = d \sin \theta$. Therefore we have

$$d \sin i + d \sin \theta = n\lambda \qquad \text{or}$$

$$\sin i + \sin \theta = \frac{n\lambda}{d} \text{*} \qquad (17)$$

in which n is any whole number.

Remark. — The line CF in Figs. 126*a* and 126*c* is normal to the incident wave fronts. The axis of the lens LL may or may not coincide with CF. In fact the axis of the lens is usually made to coincide with CF', say, if the spectrum near F' is being examined, as shown in Fig. 127.

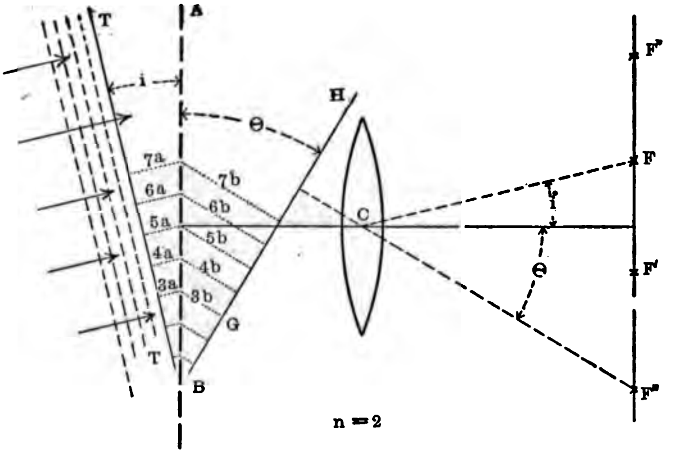

Fig. 126c.

116. Transmission gratings and reflection gratings. — Diffraction gratings for optical purposes are usually made by ruling equidistant lines on plates of glass or speculum metal. The former is called a *transmission* grating. The action of a transmission grating is discussed in Arts. 115*a* and 115*b*. The latter is called a *reflection* grating. The reflection grating consists of a number of parallel equidistant strips separated by non-reflecting spaces. The action of a reflection grating is shown in Fig. 126*d*.

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* The general form of this equation is:

$$\sin i + \sin \theta = \pm \frac{n\lambda}{d}$$

The discussion given in Art. 115*b* applies to Fig. 126*d* by changing the word *slit* to *strip.*

117. The grating spectrometer.—The diffraction grating affords an easy and a very accurate method for the determination of the wave-length of light. For this purpose the grating is mounted in place of the prism in a spectrometer (Art. 99). The arrangement of the grating spectrometer is shown in Fig. 127. A reflection grating *AB* is mounted upon an arm *R* which turns about a pivot at the center of a divided circle by means of which the angles θ and i may be measured. The light to be analyzed is admitted through the narrow slit *S* which is at the principal focus of the collimating lens *L*. After passing through *L* the light becomes an incident train of plane waves which, after striking *AB*, are focussed by the lens *L′* in a series of bright lines at *RV*. The lens *E* is an eye-piece for viewing the spectrum *RV*.

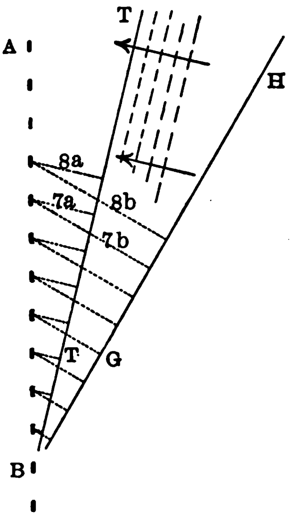

Fig. 126*d*.

The most complete and accurate table of wave-lengths is that published by H. A. Rowland in the *Astrophysical Journal* (1895–1896). The following table gives the wave-lengths of the wave-trains corresponding to the principal Fraunhofer lines in the solar spectrum as measured by Bell using a Rowland grating.

TABLE.

Line.			Line.		
A	7594.06×10^{-8}	cm.	F	4861.49×10^{-8}	cm.
B	6867.46	"	G	4308.07	"
C	6563.06	"		4307.90	"
D_1	5896.15	"	H_1	4101.85	"
D_2	5890.18	"	H_2	3968.62	"
E_1	5270.50	"			
E_2	5269.72	"			

Wave-length units. — The *micron* (symbol μ) is one millionth of a meter or one ten thousandth of a centimeter. The Ångstrom unit (symbol Å) is 10^{-8} centimeter. Wave-lengths are frequently expressed in microns, and, less frequently perhaps, in Ångstrom units.

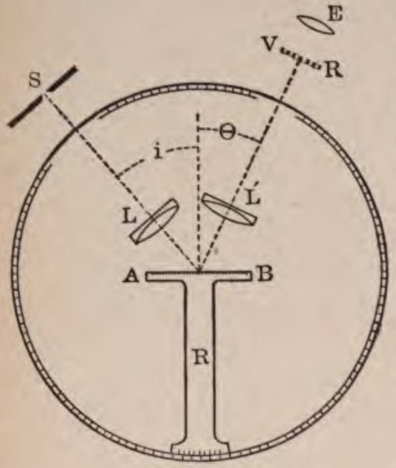

Fig. 127.

118. The concave grating. — Consider the two distances a and b, Fig. 128, of two points S and S' from a point p upon a plane or curved surface AB. As the point p moves along the surface AB the sum $a + b$ changes continuously. If only those portions of the surface AB are polished (the intervening portions being roughened by scratching) for which

$$a + b = n\lambda \qquad (18)$$

where n is an integer, and λ the wave-length of homogeneous light coming from S, then all the wavelets that emanate from the polished portions of AB will be in like phase at S', producing intense illumination there, producing, in fact an image of S at S'. Rowland has realized the condition expressed in equation (18) in his *concave grating*. This concave grating consists of a polished spherical surface of speculum metal AB, Fig. 128, ruled with lines which are the intersections of a series of equi-distant parallel planes with the curved speculum metal surface. The center of curvature of the grating surface is at C and the dotted line is a circle on Cp as a diameter. S is a slit through which light shines on the grating and the grating forms near S' distinct images of the slit one for each wave-length.

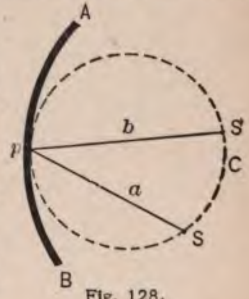

Fig. 128.

These images of the slit may be directly observed by means of an eye-piece or they may be allowed to fall upon a photographic plate bent to conform to the dotted circle.

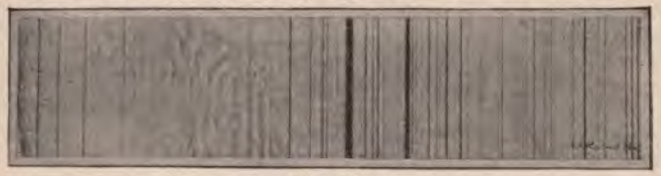

Fig. 129.

Fig. 129 is a copy of a photograph, taken in this way, of a short portion of the solar spectrum near the two D lines of Fraunhofer.

CHAPTER IX.

PHOTOMETRY.

119. Simple photometry and spectrophotometry.—The measurement of the light given out from a light source, such as a lamp, is called *photometry*. The comparison of the light from a given source, wave-length by wave-length, with the light from a standard source is called *spectro-photometry*. The comparison of the total light from a given source with the total light from a standard source is called *simple photometry*. Simple photometry is feasible only when the given light source and the standard light source are nearly or quite the same in color.

120. Light standards. *The standard candle.*—The improved illuminants which were introduced into service during the nineteenth century replaced the old-fashioned candle. It is natural that the effectiveness of the new illuminants should have been, from the first, expressed in terms of the candle, and certain specifications were long ago adopted in Great Britain, Germany, and the United States as to what should constitute the *standard candle*. These specifications remain in force to the present day as the legal definitions of the light unit, although it is now known that the candle, even when it meets the legal specifications, is one of the most variable of artificial light sources. The *British standard candle* is a sperm candle, weighing six to the pound, and burning 120 grains of sperm per hour. The *German standard candle* is made of paraffin, it is two centimeters in diameter, and it has a standard size of wick. This candle is a legal unit of light when its flame is precisely five centimeters in height.

By means of a bolometer (see Chapter XII.) and a sensitive galvanometer, it is possible to follow the continual fluctuations of brightness of a candle flame from moment to moment, and to draw a curve showing graphically the nature of these variations. Extensive measurements of this kind have been made by Sharp and Turnbull.* Fig. 130 gives a portion of such a curve plotted from measurements upon a British standard candle. The interval included in the diagram is half an hour. It will be seen that there are numerous and very sudden diminutions in intensity.

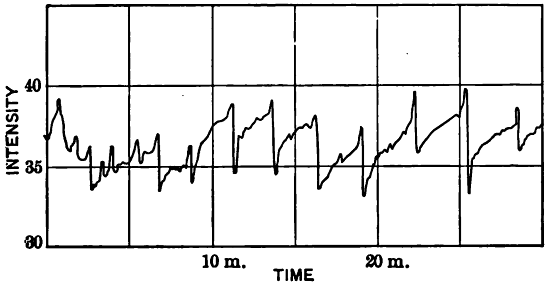

Fig. 130.—The British candle.

Various other flames and light sources have been used as standards. One of the earliest of these was the *Carcel lamp*, which was invented in France. It is a lamp with a cylindrical Argand burner and central draught. The draught is maintained by clockwork. The fuel used in this lamp is colza oil.

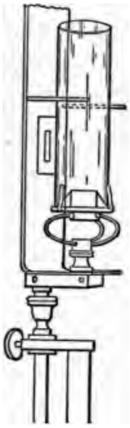

Fig. 131.

Another standard which has been extensively used is the *Methven screen.* In this apparatus an ordinary Argand gas burner is provided with an opaque screen (Fig. 131), through which a rectangular opening is cut of such size as to reduce the flame to two candle power. The opening transmits light only from the central and brightest portions of the gas flame. The variations of such a standard are those due to the varying qualities of the illuminating gas, and to the varying pressures under which it is consumed. Fig. 132 shows the curve obtained with this lamp. It will be seen that the flame was subject to many large and rapid variations.

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* *Physical Review*, Vol. II., p. 1.

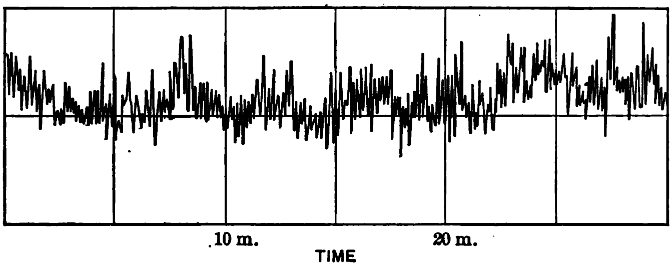

Fig. 132.—The Methven standard.

In Germany the candle has been largely supplanted as a standard by the *Hefner-Alteneck lamp*. This is a simple metal lamp, burning amyl acetate. It is arranged so that the height of the flame can be accurately adjusted and measured. It has been found that this standard is reliable to within two per cent., which is a better result than has been obtained, thus far, with other flames. Fig. 133 shows a typical bolometric curve for the Hefner lamp. During the half hour included in this test, the brightness of the flame was very nearly constant.

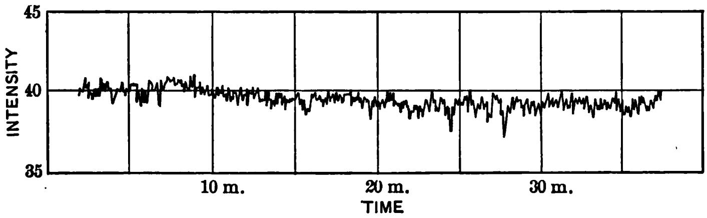

Fig. 133.—The Hefner lamp.

The glow lamp.—The electric glow lamp, when supplied with constant current, is free from the difficulties which interfere with the constancy of flame standards. It has been widely adopted in photometric work as a comparison standard, but thus far it has been found impossible to produce glow lamps which do not change in brightness with age. It has not been found practicable to so specify the character of the parts of a glow lamp that one could be manufactured which, when provided with a given current, would give a specified standard candle power.

121. The law of inverse squares. — Imagine a spherical surface of radius d described with a light source at its center. All the light from the source is distributed over this spherical surface. If the radius d is doubled the area of the spherical surface will be quadrupled and the *light per unit area* or the *intensity of illumination* of the spherical surface will be one quarter as great. Therefore *the intensity of illumination produced by a light source is inversely proportional to the square of the distance of the illuminated surface from the source.*

122. Bouguer's principle. — It was first shown by Bouguer (1726) that the relative amounts of light given out by two lamps might be determined by measuring the respective distances at which the lamps give equal intensity of illumination on a screen. This principle is the basis of most practical photometric methods.

The intensity of illumination I produced on a screen by a lamp is proportional to the total amount of light L given out by the lamp and inversely proportional to the square of the distance d of the lamp from the screen. That is:

$$I = k\frac{L}{d^2} \tag{19}$$

where k is a constant. If we have another lamp giving off an amount of light L' and producing the same intensity of illumination I at distance d' then:

$$I = k\frac{L'}{d'^2} \tag{20}$$

From these two equations, eliminating I, we have:

$$L' = \left(\frac{d'}{d}\right)^2 \cdot L \tag{21}$$

If the value, or candle power, of L is known, and if d and d' have been determined by trial, the value of L' may be calculated.

123. The shadow photometer was devised by Bouguer, and was used later in a modified form by Lambert, who wrote an extended treatise on photometry in 1760. It was also used by Rumford, and is often called *Rumford's Photometer.* The shadow photometer is an arrangement in which the two sources of light to be compared cast overlapping shadows upon a screen. as shown by the overlapping squares, A and B, in Fig. 134, The portion C of the screen which is common to both shadows is illuminated by neither source, while the remaining portions of A and B are illuminated each by a single source. The lamps are moved to such distances from the screen as to bring the contiguous portions of A and B, namely the portions d and e, to equal brightness. The distances of the lamps from the screen are measured, and the light L' given out by the one lamp may then be calculated in terms of the light L given out by the other, using equation (21).

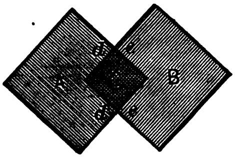

Fig. 134.

124. The Bunsen photometer.—This instrument, which has been more widely used in practical photometry than any other, depends upon Bouguer's principle, and is peculiar in the method employed in judging the equality of illumination produced upon the screen by the lamps to be compared.

The lamps, L and L', Fig. 135, to be compared are placed at the ends of a long bar upon which is a sliding carriage CC. The screen S is supported on this carriage, and one lamp illuminates one side of the screen and the other lamp illuminates the other side. The screen is made of unsized paper with an oiled spot upon it. When the illumination on the two sides of the screen is the same the oil spot disappears more or less completely according to the care which has been taken in selecting the paper for been found impossible to produce glow lamps which do not change in brightness with age. It has not been found practicable to so specify the character of the parts of a glow lamp that one could be manufactured which, when provided with a given current, would give a specified standard candle power.

121. The law of inverse squares.—Imagine a spherical surface of radius d described with a light source at its center. All the light from the source is distributed over this spherical surface. If the radius d is doubled the area of the spherical surface will be quadrupled and the *light per unit area* or the *intensity of illumination* of the spherical surface will be one quarter as great. Therefore *the intensity of illumination produced by a light source is inversely proportional to the square of the distance of the illuminated surface from the source.*

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$$I = k\frac{L'}{d'^2} \qquad (20)$$

From these two equations, eliminating I, we have:

$$L' = \left(\frac{d'}{d}\right)^2 \cdot L \qquad (21)$$

of the field the left hand side of the screen is visible. Portions of the two sides of the screen being thus seen in close juxtaposition, very slight differences in brightness may be detected.

126. Intensity of Illumination. Intrinsic Brightness. — By *one candle* is meant the total light emanating from a standard candle.* The *intensity of illumination* of a surface is defined as the total number of candles of light falling upon the surface, divided by the area of the surface. Thus the intensity of illumination at a distance of 100 cm. from a standard candle is about 0.000008 candle per square centimeter. The least intensity of illumination required for comfortable reading is about four times as great as this, or 0.000032 candle per square centimeter.

The *intrinsic brightness* of a luminous body is defined as the total number of candles of light given out by the body, divided by the area of its shining surface. Thus the area of the flame of a standard candle is about 10 square centimeters, and its intrinsic brightness is 0.1 candle per square centimeter. The intrinsic brightness of an Argand gas flame is about one candle per square centimeter. The intrinsic brightness of the filament of an electric glow lamp is from 10 to 20 candles per square centimeter. The intrinsic brightness of the crater of an arc lamp is as much as 20,000 candles per square centimeter.

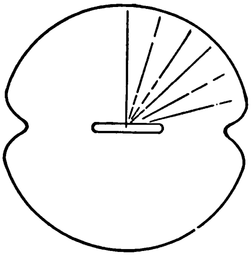

Fig. 136.

127. Distribution of light around a lamp. —A lamp usually gives out light unequally in different directions. Thus the radii vectores of the curve, Fig. 136, represent the intensities of light given out by the flame of a kerosene lamp in different directions in a horizontal plane. There is slightly less light given off by the edge than by the broad side of the flame. The distribution of light from an arc lamp is very far from uniform. In commercial direct-current arc lamps the current is always sent through the lamp in such a direction that the crater is formed at the end of the upper carbon as shown in Fig. 137. Most of the light from the lamp comes from this crater. The radii vectores of the curve, Fig. 137, represent the intensities of light given out by the arc lamp in different directions in a vertical plane.

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* Strictly on the assumption that the candle gives out in all directions light of the same intensity as that given out in a horizontal plane.

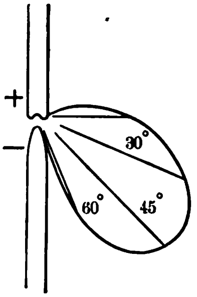

Fig. 137.

The distribution of light around a lamp is determined by placing the lamp upon the photometer bar, turning the lamp into various positions, and reading the photometer for each position. In case the lamp cannot be readily turned, systems of mirrors are used to reflect light, coming in different directions from the lamp, along the photometer bar.

The *mean spherical candle power*, or true candle power, of a lamp is the total number of candles of light given out by the lamp. The candle power of a lamp as measured by a photometer may be far above or below the true candle power according as the lamp when placed upon the photometer bar is turned so as to send a large or a small amount of light to the photometer screen. The true candle power may be determined by properly* averaging the photometer readings for many different positions of the lamp.

128. The flicker photometer is a device for eliminating to some extent the error in setting a photometer due to differences of color of the lamps which are compared. The following is the principle upon which the elimination of color errors is based: When we look at a thing such as a photometer screen, we have a sensation of *brightness* and a sensation of *color*. Both of these sensations persist for an appreciable interval of time after light ceases to enter the eye, but *sensations of color persist longer than*

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*The separate readings must refer to solid angles, or cones, of the same size; or if they do not, each reading must be multiplied by the spherical area to which it refers and the sum of such products must be divided by the total spherical area.

sensations of brightness. If the two sides of the photometer screen are brought into the same field of view in rapid succession, a high frequency of interchange will completely blend the color sensations produced by the two sides of the screen and also the brightness sensations produced by the two sides of the screen. A much lower frequency of interchange will suffice to blend the color sensations but leave a flickering sensation of brightness unless the two sides of the screen have the same brightness. This fact was discovered by Rood.

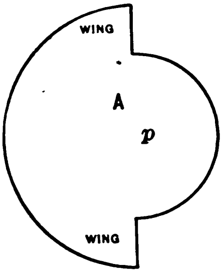

Fig. 138.

The device employed by Whitman in his flicker photometer is as follows: A white cardboard disk *A*, Figs. 138 and 139, is mounted on the axis *pp* and a stationary piece of the same cardboard is placed at *B*, Fig. 139. When the disk *A* is rotated the wing of the disk and the cardboard *B* are seen in succession by the eye placed at *e*. The wing of the disk is illuminated by the source S_1 and the cardboard *B* is illuminated by the source S_2. The disk *A* is driven at sufficient speed to blend the color sensations and then the distances of the two lamps S_1 and S_2 are adjusted until the flickering sensation of brightness disappears.*

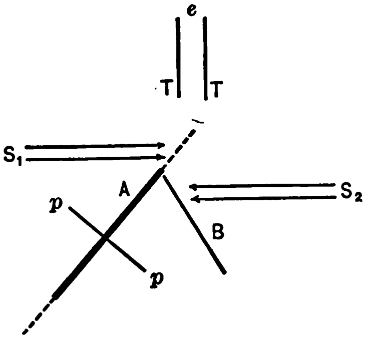

Fig. 139.

129. The spectrophotometer is a combined spectroscope and photometer for comparing the intensities of two light sources, wave-length by wave-length. Fig. 140 shows a lamp *L* and a light standard placed at ends of a photometer bar. The heavy black line *S* represents the slit of a spectroscope. In front of this slit are two total reflecting prisms *pp*. One of these prisms reflects light from L into one end of the slit and the other reflects light from the standard into the other end of the slit, so that the spectra of the two lights are seen side by side in the spectroscope. Fig. 141 shows the spectroscope (direct-vision variety) mounted on a car so that it may be easily moved along the photometer bar, thus changing the relative intensity of the two spectra at will. Observations are taken with this instrument as follows: The observer's attention is fixed upon a certain region of the two spectra, say the extreme red, and the car is moved until the two spectra are of the same intensity in this region. The ratio of brightness of the two lights for this region of the spectrum is then equal to $(l'/l)^2$ where l and l' are the distances shown in Fig. 140. This operation is repeated throughout the spectrum.

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* F. P. Whitman, *Physical Review*, Vol. III., p. 241.

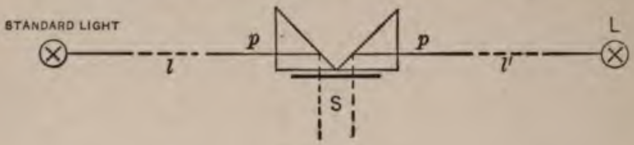

Fig. 140.

There are many other devices by means of which the two spectra to be compared in spectrophotometry may be reduced to equal brightness. Sometimes the widths of the two ends of the slit are varied; sometimes a wedge of neutrally tinted glass is interposed in the path of the brighter beam; sometimes a pair of Nicol prisms (see the chapter on Polarization) is used.

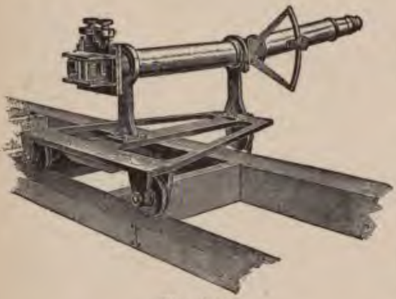

Fig. 141.

130. Examples of spectrophotometric measurements.—The results of a spectrophotometric determination are chiefly interesting as showing the relative intensities of the various parts of the spectrum of one light compared with another light without regard to the total intensity of either light. Thus we may choose any light as a standard of comparison, say gaslight. In comparing another given light with this, we may make the other light of such intensity as to reduce the two spectra to equal intensity in the region of the *D* line. Then we may express the intensity of any other part of the spectrum of the given light in terms of the intensity of the corresponding part of the spectrum of gaslight.

The curves in Fig. 142 show the results of a spectrophotometric comparison of daylight and limelight with gaslight. The composition of the gaslight is represented by a horizontal straight line, inasmuch as the intensity at each part of the spectrum of gaslight is taken as the *unit* intensity for that part of the spectrum. The daylight and limelight are each of such total intensity as to have the same intensity as the gaslight in the region of the *D* line (wave-length 0.59μ). The ordinate of the daylight (or limelight) curve at any point represents the

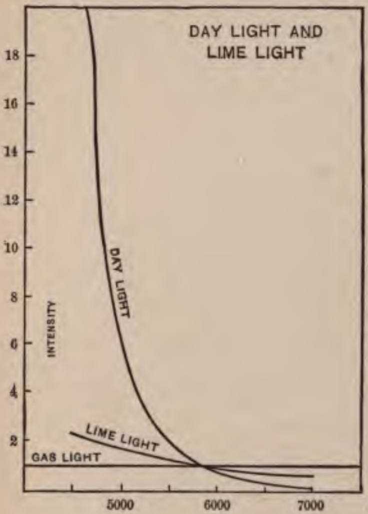

Fig. 142.

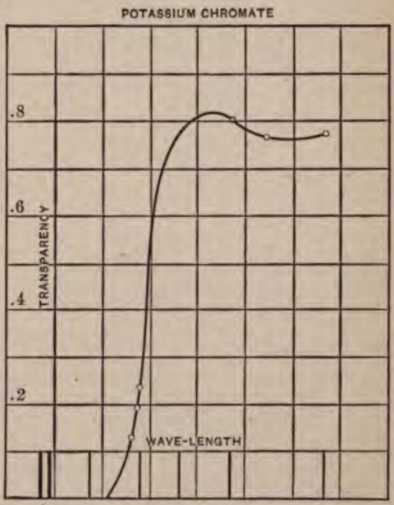

Fig. 143.

number of times brighter the daylight (or limelight) spectrum is than the gaslight spectrum at that point. Thus, daylight, which has the same intensity as gaslight at the *D* line, is only about $\frac{1}{10}$ as bright as the gaslight at wave-length 7,000, and more than twenty times as bright as gaslight at wave-length 4,500.

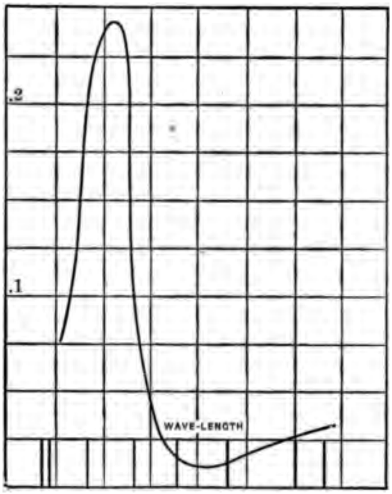

Fig. 144.

The curves in Figs. 143 and 144 show the results of a spectrophotometric study of gaslight which has passed through a solution of potassium chromate, and of gaslight which has been reflected from powdered ultramarine blue, both referred to gaslight direct. The ordinates in Fig. 143 show the fractional parts of the various wave-lengths of gaslight which are transmitted by the solution and the ordinates in Fig. 144 show the fractional parts of the various wave-lengths of gaslight which are reflected by ultramarine blue. Thus of wave-length 6,000 nearly eight tenths is transmitted by the potassium bichromate solution, and about one eightieth is reflected by the ultramarine blue.

The Composition of Light. — We shall frequently speak of the *composition* of light, especially in the chapter on color and in the chapter on radiation, meaning simply the relative intensities of the various homogeneous components (wave-lengths) of the light.

CHAPTER X.

COLOR.

131. Sensations of brightness and of color. — The special sensations due to light are *brightness* and *color*. Brightness depends upon the intensity of the light, and color depends upon the wave-length * or, when the light is not homogeneous, upon the relative intensity of the various wave-lengths. We shall, in the following discussion, consider first the sensations of brightness and of color which are produced by the various parts of the spectrum, that is to say, the sensations of brightness and of color which are produced by various wave-lengths of homogeneous light. Then we shall consider the sensations of color which are produced by mixed light, or light which is composed of many wave-lengths.

132. Brightness of homogeneous light. — The various parts of the spectrum, of gaslight for example, differ greatly in brightness and by no means in proportion to the energy that is present in the various parts of the spectrum. Thus the ordinates of the dotted curve in Fig. 145 represent the energy intensity or heating effect of the various parts of the spectrum of gaslight, and the ordinates of the full line curve represent the brightness of the various parts of the spectrum of gaslight. This curve of brightness may be determined by the flicker photometer (see Art. 128) as follows: One screen *A* of the flicker photometer is illuminated by a definite part of the gaslight spectrum. The other screen *B* of the photometer is illuminated by a constant light source of which the distance from the screen is adjusted until the two screens are of the same brightness as indicated by the cessation of the flicker. The screen *A* is then illuminated by another part of the spectrum and the same adjustment is repeated, and so on. Then the intensities (ordinates of the curve) of the various parts of the gaslight spectrum are to each other inversely as the squares of the corresponding distances of the constant light source.

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* This statement assumes that the light is trave [illegible] eous medium where the velocity is constant. On passing [illegible] air to glass, the velocity is diminished, and conse [illegible] frequency, however, remains unchanged. [illegible] color when passing from air to glass, c [illegible] than on wave-length.

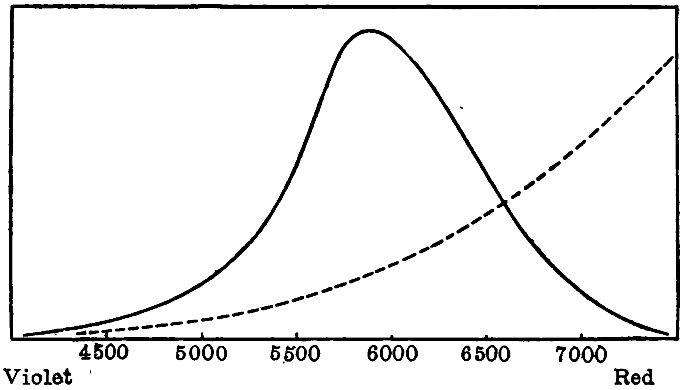

Fig. 145.

The ordinates of the energy curve in Fig. 145 were determined by means of the bolometer as explained in Chapter XII.

133. Color sensations due to homogeneous light.—The various wave-lengths of homogeneous light produce a variety of color sensations. Newton recognized and named seven colors in the spectrum: red, orange, yellow, green, blue, indigo, and violet. About one hundred and fifty steps are made, however, in going through the spectrum from one tint to the next, which can barely be distinguished from it. That is to say, there are about one hundred and fifty distinguishable tints in the spectrum. The most vivid colors in the spectrum are the extreme red, the green, and the violet.

134. Color sensations due to mixed light.—It is evident that there is a possibility of an infinite variety of mixtures of the various wave-lengths of light according to the relative intensities of the various wave-lengths that are present in the mixture. The number of distinguishable color sensations which may be produced by the various light mixtures is, according to Titchener,* about thirty thousand, and according to Rood † a much larger number.

White light.—Sunlight, or any light approaching sunlight in composition, is called white light. The sensation produced by such light, aside from complications growing out of contrast effects, is called white.

Saturated and diluted colors.—Mixed light which contains a great excess of one wave-length or group of wave-lengths generally produces a vivid sensation of color. Such a color is sometimes called a *saturated* color. Mixed light which approaches white light in composition, having only a slight excess of one wave-length or group of wave-lengths, gives a pale sensation of color. Such a color is sometimes called a *diluted* color.

135. The cause of color in natural objects.—Colored objects occurring in nature owe their color to the fact that they send to the eye light of which the composition differs more or less widely from the composition of white light. Mixed light differing from white light in composition may be produced by selective radiation, by selective reflection, by selective transmission, by dispersion, and by interference. Selective radiation, reflection, and transmission are discussed in Chapter XII. Dispersion is discussed in Chapter VII. Interference is discussed in Chapter VIII.

Examples.—(*a*) Hot gases and vapors give off light which differs widely from white light in composition. Thus most of the color effects in fireworks are produced by the use of salts of various metals such as strontium and copper. These salts are vaporized and the hot vapors give off brilliantly colored light.

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* Titchener, "An Outline of Psychology," p. 66.

† Rood, "Modern C[illegible]." Chapter IX.

(*b*) Many substances reflect or transmit in excess certain wave-lengths of the light which falls upon them. Two examples of this are described in Art. 130. Such substances appear colored when illuminated by white light. When such substances are illuminated by yellowish gaslight their colors appear less brilliant, and when illuminated by homogeneous light, such as the light from a sodium flame, all color differences disappear.

136. Color mixing. Dichroic and Trichroic Vision. — Consider two (or more) beams of light of different composition. These beams give distinctly different sensations of color. If they are mixed * (*i. e.*, allowed to enter the eye and fall upon the same portion of the retina), the sensation is still that of a single definite color.

Any color may be matched by properly mixing a saturated red light, a saturated green light, and a saturated violet light. To this end, the intensity of each colored light must be under control and varied until the mixture matches the given color. This is an experimental fact first pointed out by Thomas Young.

For some persons (red-blind) any color may be matched by properly mixing a saturated green light and a saturated violet light. For other persons (green-blind) any color may be matched by properly mixing a saturated red light and a saturated violet light.

Persons for whom any color may be matched by mixing two saturated colors are said to have *dichroic* vision. Persons for whom the matching of any color requires the mixing of three saturated colors are said to have *trichroic* vision. Dichroic vision is commonly called *color-blindness.* About four per cent. of the male population and about four out of every thousand of the female population of the civilized world are color-blind.

The color top. — The most convenient arrangement for mixing colored light is by means of the color top. This consists of a rotating spindle which carries three colored circular disks (red, green and blue) slitted in such a way that any desired sector of the face of each disk may be exposed to view. When this triple disk is rotated rapidly, the colors blend and give a single sensation of color, the tint of which may be modified at will by varying the amounts of exposure of the respective disks.

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* The mixing of *colored lights* and the mixing of *pigments* are essentially different in the results produced as is explained in Art. 165.

Complementary colors. — Two colors which produce white light when mixed are called complementary colors.

137. The Young-Helmholtz theory of color. — The experimental fact stated in the previous article led Thomas Young (1801) to infer the existence of three *primary color sensations.* Helmholtz attributed each of these primary sensations to a distinct set of nerves in the retina of the eye. The nerves which, upon excitation, give the primary sensation of red are called the *red nerves;* those which, upon excitation, give the primary sensation of green are called the *green nerves;* and the set which, upon excitation, gives the primary sensation of violet, the *violet nerves.* Simultaneous excitation of all three sets of nerves gives a *blended sensation*, the character of which depends upon the degrees of excitation of the respective sets of nerves.

A person having dichroic vision has only two sets of color nerves, and, therefore, only two primary color sensations. Persons who do not have the primary sensation of red are said to be *red-blind;* persons who do not have the primary sensation of green are said to be *green-blind.*

The Young-Helmholtz theory of color is not the only theory in vogue; indeed, physiologists are inclined to reject it for lack of microscopical evidence of the existence of the three sets of nerves; and psychologists are inclined to reject it, mainly because of the difficulty of explaining the great number of distinguishable color sensations by the varying intensities of sensation on three sets of nerves. The theory gives, however, a very clear representation of the experimental facts stated in the previous article, and a very satisfactory explanation of contrast effects and of color-blindness.

Sensitiveness of the color nerves to different wave-lengths. — When a given sensory nerve is excited, by whatever means, the sensation is always the same; thus, excitation of the *red*, ***green***, or *violet* nerves gives sensations of red, green, or violet respectively, however the excitation be produced,

The ordinates of the three curves in Fig. 146 represent the relative degrees of sensitiveness of the three sets of nerves to the various wave-lengths of light. These curves are from measurements made by Koenig. Thus every wave-length in the spectrum is capable of affecting the red nerves and in some degree producing the sensation of red. The green nerves are more or less affected by all wave-lengths between the Fraunhofer lines *C* and *G*, and the violet nerves are more or less affected by all wave-lengths between the Fraunhofer lines *E* and *H*. Furthermore, a given wave-length or mixture of wave-lengths generally affects all three sets of nerves, giving a blended sensation of color.

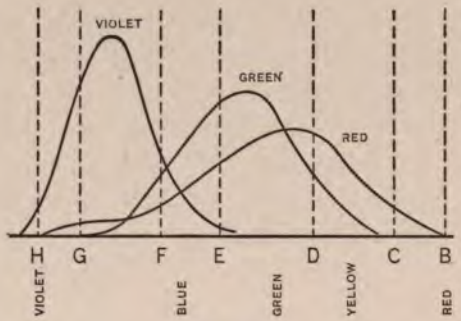

Fig. 146.

138. Contrast effects. — Two complementary colors viewed side by side tend to intensify each other mutually as the eye wanders from the one to the other. Any two colors viewed side by side, if they are at all different, tend to become complementary in appearance as the eye wanders from one to the other. This contrast effect, as it is called, may be shown very strikingly by taking two small pieces of green paper exactly alike and placing one on a large sheet of red paper and the other on a large sheet of blue paper. The two pieces of green paper are so

greatly changed by the opposite contrasts that one can scarcely believe that they are physically alike. A bit of white or grayish paper placed upon a broad sheet of brilliantly colored paper assumes very distinctly a hue complementary to the surrounding color.

The explanation of contrast effects in terms of the Young-Helmholtz theory of color is as follows: White light affects all three sets of nerves approximately equally. A colored light affects all three sets of nerves and the vividness of the color sensation produced depends upon the preponderating excitation of one or two sets of nerves. When two colored lights A and B are complementary then those nerves which are most affected by A are least affected by B. When one looks intently at a brilliantly colored light the one or two sets of nerves which are most strongly excited become fatigued. If one then looks at another light these fatigued nerves are less excited than they would be under normal conditions and the second colored light tends to assume a hue which is complementary to the first.

139. Dichroic vision. — Color-blind persons show marked peculiarities in their sensations of brightness and of color.

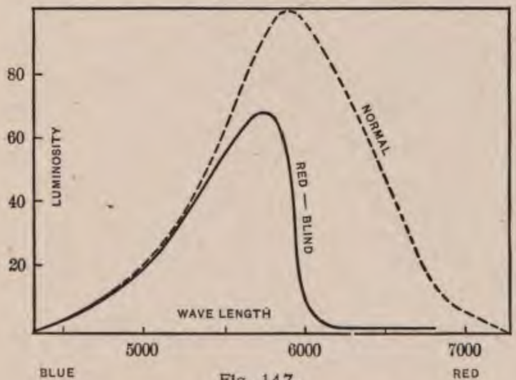

Fig. 147.

Brightness of homogeneous light to color-blind persons. — The ordinates to the dotted curve in Fig. 147 represent the brightness of the various parts of the spectrum of gaslight to a person

who has normal trichroic vision and the ordinates of the dotted curve* represent the brightness of the various parts of the same spectrum to a person who is red-blind. No data are available concerning the brightness of the various parts of the spectrum to a person who is green-blind.

Color sensations due to homogeneous light.—Fig. 148 shows roughly the appearance of the spectrum to persons with normal trichroic vision, to green-blind persons and to red-blind persons. The red and violet nerves of the green-blind person are both affected more or less by any wave-length or mixture of wave-lengths and that particular wave-length in the spectrum that affects these two nerves equally, as ordinary white light affects them equally, gives the sensation called white by the green-blind person. This wave-length lies between the E and F lines according to Fig. 146.

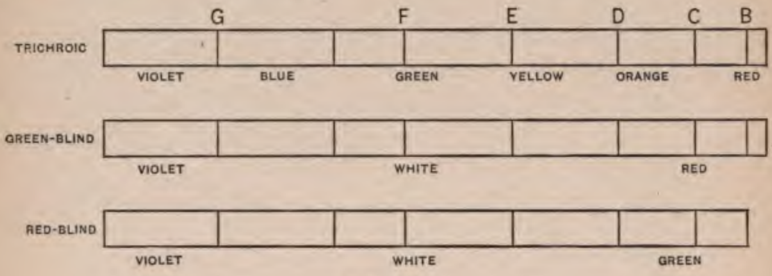

Fig. 148.

The green and violet nerves of the red-blind person are both affected more or less by any wave-length or mixture of wave-lengths, and that particular wave-length (also between the E and F lines according to Fig. 146) which affects these two nerves equally gives the sensation called white by the red-blind person. The red end of the spectrum does not extend much beyond the C line for the red-blind person.

Color sensations due to mixed light.—Color-blind persons show marked peculiarities of color sensation due to mixed

lights. These peculiarities are, of course, very complicated; in fact they are hardly the same for any two persons. The following article which describes the Holmgren test for color-blindness gives some idea of the peculiar color sensations of the color-blind persons produced by mixed light.

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* From measurements by Ferry, *American Journal of Science*, Vol. 44 (1892).

140. The Holmgren test for color-blindness.—Since all our systems of signaling, both upon railways and at sea, depend upon the recognition of colored lights, the character of the color vision of employees is a matter of the utmost practical importance. It is fortunately possible to detect dichroic vision with certainty even where the existence of the peculiarity is unsuspected by the subject himself. The simplest satisfactory method of performing such tests was invented by Holmgren of Upsala, after the occurrence of a terrible railway accident due to the color-blindness of an employee. The Holmgren apparatus consists simply of a collection of colored worsteds, which includes a large variety of colors of every degree of saturation and a number of neutral grays of various degrees of brightness. There are also three skeins, called the confusion samples. One of these is a pale green, corresponding very nearly in tint (but of course not in saturation) to the portion of the spectrum which appears white to both red-blind and green-blind persons as described in Art. 139. Another confusion sample is a pale magenta, which is a blend of red and violet with much white, and the third confusion sample is a brilliant red.

Both red-blind and green-blind persons invariably select a variety of neutral or gray worsteds as corresponding most closely with the pale green confusion sample.

Red-blind persons invariably select violets and blues as corresponding most closely with the magenta sample.

Both red-blind and green-blind persons are inclined to select brilliant greens as well as brilliant reds in matching the bright red confusion sample.

The selections made by color-blind persons, when attempting to follow their own judgment in matching these confusion sam-

ples in the Holmgren test, are very striking to an observer with normal vision. A university student, who was red-blind, but who showed great power of discrimination in his choice of colors, as viewed from his own standard, and who was fully aware of the nature of his color sense, made the following selections of the worsteds which seemed to him most nearly related to the confusion samples.

Of the twelve skeins of worsted selected to go with the pale green sample, only one contained any green pigment. The remainder were very pale yellows and browns, almost without coloring matter, and two skeins of very pale rose color.

All the worsteds selected on account of their resemblance to the magenta sample were purples in which blue predominated, or nearly pure blues.

With the bright red sample were placed a few other nearly pure reds, together with a much larger number of dark browns and greens.

Surprising as these selections seem to one accustomed to the trichroic color system, they are all readily accounted for under the Young-Helmholtz theory of colors.

CHAPTER XI.

POLARIZATION AND DOUBLE REFRACTION.

141. Polarization of transverse waves. — Transverse waves are said to be *plane polarized* or simply *polarized* when the oscillations are all in one direction or in one plane. Transverse waves are said to be *circularly polarized* when a particle of the transmitting medium describes a circular path as the waves pass by it. Transverse waves are said to be *elliptically polarized* when a particle of the transmitting medium describes an elliptical path as the waves pass by it.

These special types of transverse waves are most clearly understood by a simple study of the transverse waves produced on a stretched rubber tube one end of which is held in the hand. If the hand is moved rapidly up and down a train of plane polarized waves is produced, the oscillations all along the rubber tube being in a vertical plane. If the hand is moved rapidly in a small circular path, around the stretched tube as an axis, a train of circularly polarized waves is produced and any given particle of the tube describes a circular path as the wave-train passes by it. If the hand is moved rapidly in a small elliptical path a train of elliptically polarized waves is produced, and any given particle of the tube describes an elliptical path as the train of waves passes by it.

Ordinary light. — When one end of the stretched rubber tube used in the above illustration is moved up and down, then to and fro sidewise, then in a circular or elliptical path in irregular succession an irregular series of waves will pass along the tube. Ordinary light, such as comes from a luminous body, consists of an irregular series of waves very similar to these irregular waves

on the rubber tube. Ordinary light is not characterized by any persistent and simple type of oscillation.

A plane polarized series of waves may be separated out from an irregular series of waves, that is from a series devoid of any persistent and simple type of oscillation, by stretching the rubber tube through a narrow slit in a board. Those waves of which the oscillations are parallel to the slit will pass through the slit freely; those waves of which the oscillations are at right angles to the slit will not pass through at all, but will be turned back or reflected; those waves of which the oscillations are inclined to the slit will be partly transmitted and partly reflected, the oscillations will be resolved by the slit into two components *parallel* to the slit and *perpendicular* to the slit respectively, the former component-oscillations will pass through the slit and the latter component-oscillations will be turned back by the slit or reflected. The slit will thus resolve the irregular series of waves into two plane polarized series of waves with their planes of oscillation at right angles to each other, one series being transmitted and the other turned back or reflected.

If the rubber tube is stretched through two slits the waves which are transmitted by the first slit will pass freely through the second slit if the slits are parallel, the waves will be partially transmitted by the second slit if the slits are inclined to each other, and the waves will be completely stopped by the second slit if the slits are at right angles to each other.

In longitudinal waves the oscillations are always in the direction of progression of the waves and, therefore, there can be no special types of longitudinal waves which depend upon the direction or path of the oscillations. The phenomena of polarization can be exhibited by transverse waves only.

142. The optical behavior of tourmaline crystals; polarized light. — A plate of tourmaline, cut parallel to the axis of the crystal, transmits only a portion of the light which falls upon it. The light which passes through such a plate passes freely through a similar plate when the axes of the plates are parallel, and is shut off entirely when the axes of the plates are at right angles. The beam of light transmitted by the first plate is *polarized.* When the second plate is slowly turned about an axis parallel to the beam of light, the intensity of the transmitted beam changes slowly from maximum intensity when the axes of the plates are parallel, to zero intensity when the axes are crossed. This is shown by the shading in Fig. 149. The fact that light can be polarized shows that light waves are transverse waves.

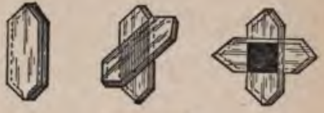

Fig. 149.

Description of ordinary light, of plane polarized light, of circularly polarized light, and of elliptically polarized light in terms of the elastic solid theory. — Imagine a large number of thin flat boards arranged side by side as shown in Fig. 150, the spaces between the boards being filled with an elastic jelly. If the board *AB* be moved rapidly up and down a *plane polarized* train of plane * waves will be transmitted to the right as indicated by the dotted arrows, and each board will oscillate up and down as the wave-train passes by it.

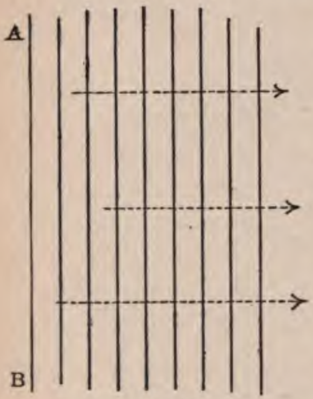

Fig. 150.

If the board *AB* is moved so that each point in the board *AB* describes a small circular or elliptical path in the plane of the board a *circularly* or *elliptically polarized* train of plane waves will be transmitted to the right as indicated by the dotted arrows, and each board will oscillate in a manner similar to the original oscillations of *AB* as the wave-train passes by.

This motion of the system of boards represents the motion of the luminiferous ether in a beam of parallel rays (plane waves) sensation is always the same; thus, excitation of the *red*, ***green***, or *violet* nerves gives sensations of red, green, or violet respectively, however the excitation be produced,

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* Waves with plane fronts.

The ordinates of the three curves in Fig. 146 represent the relative degrees of sensitiveness of the three sets of nerves to the various wave-lengths of light. These curves are from measurements made by Koenig. Thus every wave-length in the spectrum is capable of affecting the red nerves and in some degree producing the sensation of red. The green nerves are more or less affected by all wave-lengths between the Fraunhofer lines *C* and *G*, and the violet nerves are more or less affected by all wave-lengths between the Fraunhofer lines *E* and *H*. Furthermore, a given wave-length or mixture of wave-lengths generally affects all three sets of nerves, giving a blended sensation of color.

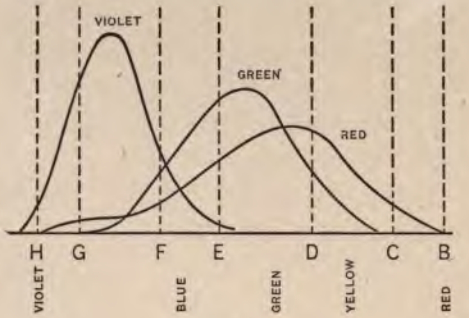

Fig. 146.

138. Contrast effects. — Two complementary colors viewed side by side tend to intensify each other mutually as the eye wanders from the one to the other. Any two colors viewed side by side, if they are at all different, tend to become complementary in appearance as the eye wanders from one to the other. This contrast effect, as it is called, may be shown very strikingly by taking two small pieces of green paper exactly alike and placing one on a large sheet of red paper and the other on a large sheet of blue paper. The two pieces of green paper are so

greatly changed by the opposite contrasts that one can scarcely believe that they are physically alike. A bit of white or grayish paper placed upon a broad sheet of brilliantly colored paper assumes very distinctly a hue complementary to the surrounding color.

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139. Dichroic vision. — Color-blind persons show marked peculiarities in their sensations of brightness and of color.

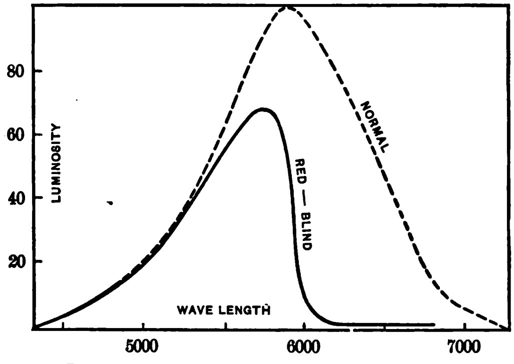

Fig. 147.

Brightness of homogeneous light to color-blind persons. — The ordinates to the dotted curve in Fig. 147 represent the brightness of the various parts of the spectrum of gaslight to a person

who has normal trichroic vision and the ordinates of the dotted curve* represent the brightness of the various parts of the same spectrum to a person who is red-blind. No data are available concerning the brightness of the various parts of the spectrum to a person who is green-blind.

Color sensations due to homogeneous light.—Fig. 148 shows roughly the appearance of the spectrum to persons with normal trichroic vision, to green-blind persons and to red-blind persons. The red and violet nerves of the green-blind person are both affected more or less by any wave-length or mixture of wave-lengths and that particular wave-length in the spectrum that affects these two nerves equally, as ordinary white light affects them equally, gives the sensation called white by the green-blind person. This wave-length lies between the E and F lines according to Fig. 146.

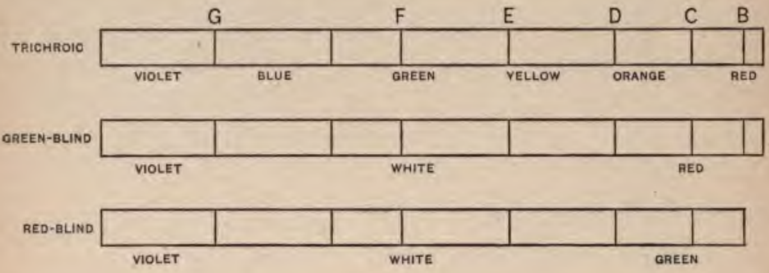

Fig. 148.

The green and violet nerves of the red-blind person are both affected more or less by any wave-length or mixture of wave-lengths, and that particular wave-length (also between the E and F lines according to Fig. 146) which affects these two nerves equally gives the sensation called white by the red-blind person. The red end of the spectrum does not extend much beyond the C line for the red-blind person.

Color sensations due to mixed light.—Color-blind persons show marked peculiarities of color sensation due to mixed

lights. These peculiarities are, of course, very complicated; in fact they are hardly the same for any two persons. The following article which describes the Holmgren test for color-blindness gives some idea of the peculiar color sensations of the color-blind persons produced by mixed light.

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* From measurements by Ferry, *American Journal of Science*, Vol. 44 (1892).

140. The Holmgren test for color-blindness.—Since all our systems of signaling, both upon railways and at sea, depend upon the recognition of colored lights, the character of the color vision of employees is a matter of the utmost practical importance. It is fortunately possible to detect dichroic vision with certainty even where the existence of the peculiarity is unsuspected by the subject himself. The simplest satisfactory method of performing such tests was invented by Holmgren of Upsala, after the occurrence of a terrible railway accident due to the color-blindness of an employee. The Holmgren apparatus consists simply of a collection of colored worsteds, which includes a large variety of colors of every degree of saturation and a number of neutral grays of various degrees of brightness. There are also three skeins, called the confusion samples. One of these is a pale green, corresponding very nearly in tint (but of course not in saturation) to the portion of the spectrum which appears white to both red-blind and green-blind persons as described in Art. 139. Another confusion sample is a pale magenta, which is a blend of red and violet with much white, and the third confusion sample is a brilliant red.

Both red-blind and green-blind persons invariably select a variety of neutral or gray worsteds as corresponding most closely with the pale green confusion sample.

Red-blind persons invariably select violets and blues as corresponding most closely with the magenta sample.

Both red-blind and green-blind persons are inclined to select brilliant greens as well as brilliant reds in matching the bright red confusion sample.

The selections made by color-blind persons, when attempting to follow their own judgment in matching these confusion sam-

ples in the Holmgren test, are very striking to an observer with normal vision. A university student, who was red-blind, but who showed great power of discrimination in his choice of colors, as viewed from his own standard, and who was fully aware of the nature of his color sense, made the following selections of the worsteds which seemed to him most nearly related to the confusion samples.

Of the twelve skeins of worsted selected to go with the pale green sample, only one contained any green pigment. The remainder were very pale yellows and browns, almost without coloring matter, and two skeins of very pale rose color.

All the worsteds selected on account of their resemblance to the magenta sample were purples in which blue predominated, or nearly pure blues.

With the bright red sample were placed a few other nearly pure reds, together with a much larger number of dark browns and greens.

Surprising as these selections seem to one accustomed to the trichroic color system, they are all readily accounted for under the Young-Helmholtz theory of colors.

CHAPTER XI.

POLARIZATION AND DOUBLE REFRACTION.

141. Polarization of transverse waves. — Transverse waves are said to be *plane polarized* or simply *polarized* when the oscillations are all in one direction or in one plane. Transverse waves are said to be *circularly polarized* when a particle of the transmitting medium describes a circular path as the waves pass by it. Transverse waves are said to be *elliptically polarized* when a particle of the transmitting medium describes an elliptical path as the waves pass by it.

These special types of transverse waves are most clearly understood by a simple study of the transverse waves produced on a stretched rubber tube one end of which is held in the hand. If the hand is moved rapidly up and down a train of plane polarized waves is produced, the oscillations all along the rubber tube being in a vertical plane. If the hand is moved rapidly in a small circular path, around the stretched tube as an axis, a train of circularly polarized waves is produced and any given particle of the tube describes a circular path as the wave-train passes by it. If the hand is moved rapidly in a small elliptical path a train of elliptically polarized waves is produced, and any given particle of the tube describes an elliptical path as the train of waves passes by it.

Ordinary light. — When one end of the stretched rubber tube used in the above illustration is moved up and down, then to and fro sidewise, then in a circular or elliptical path in irregular succession an irregular series of waves will pass along the tube. Ordinary light, such as comes from a luminous body, consists of an irregular series of waves very similar to these irregular waves

(*b*) Many substances reflect or transmit in excess certain wave-lengths of the light which falls upon them. Two examples of this are described in Art. 130. Such substances appear colored when illuminated by white light. When such substances are illuminated by yellowish gaslight their colors appear less brilliant, and when illuminated by homogeneous light, such as the light from a sodium flame, all color differences disappear.

136. Color mixing. Dichroic and Trichroic Vision. — Consider two (or more) beams of light of different composition. These beams give distinctly different sensations of color. If they are mixed * (*i. e.*, allowed to enter the eye and fall upon the same portion of the retina), the sensation is still that of a single definite color.

Any color may be matched by properly mixing a saturated red light, a saturated green light, and a saturated violet light. To this end, the intensity of each colored light must be under control and varied until the mixture matches the given color. This is an experimental fact first pointed out by Thomas Young.

For some persons (red-blind) any color may be matched by properly mixing a saturated green light and a saturated violet light. For other persons (green-blind) any color may be matched by properly mixing a saturated red light and a saturated violet light.

Persons for whom any color may be matched by mixing two saturated colors are said to have *dichroic* vision. Persons for whom the matching of any color requires the mixing of three saturated colors are said to have *trichroic* vision. Dichroic vision is commonly called *color-blindness.* About four per cent. of the male population and about four out of every thousand of the female population of the civilized world are color-blind.

The color top. — The most convenient arrangement for mixing colored light is by means of the color top. This consists of a rotating spindle which carries three colored circular disks (red, green and blue) slitted in such a way that any desired sector of the face of each disk may be exposed to view. When this triple disk is rotated rapidly, the colors blend and give a single sensation of color, the tint of which may be modified at will by varying the amounts of exposure of the respective disks.

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* The mixing of *colored lights* and the mixing of *pigments* are essentially different in the results produced as is explained in Art. 165.

Complementary colors. — Two colors which produce white light when mixed are called complementary colors.

137. The Young-Helmholtz theory of color. — The experimental fact stated in the previous article led Thomas Young (1801) to infer the existence of three *primary color sensations.* Helmholtz attributed each of these primary sensations to a distinct set of nerves in the retina of the eye. The nerves which, upon excitation, give the primary sensation of red are called the *red nerves;* those which, upon excitation, give the primary sensation of green are called the *green nerves;* and the set which, upon excitation, gives the primary sensation of violet, the *violet nerves.* Simultaneous excitation of all three sets of nerves gives a *blended sensation*, the character of which depends upon the degrees of excitation of the respective sets of nerves.

A person having dichroic vision has only two sets of color nerves, and, therefore, only two primary color sensations. Persons who do not have the primary sensation of red are said to be *red-blind;* persons who do not have the primary sensation of green are said to be *green-blind.*

The Young-Helmholtz theory of color is not the only theory in vogue; indeed, physiologists are inclined to reject it for lack of microscopical evidence of the existence of the three sets of nerves; and psychologists are inclined to reject it, mainly because of the difficulty of explaining the great number of distinguishable color sensations by the varying intensities of sensation on three sets of nerves. The theory gives, however, a very clear representation of the experimental facts stated in the previous article, and a very satisfactory explanation of contrast effects and of color-blindness.

Sensitiveness of the color nerves to different wave-lengths. — When a given sensory nerve is excited, by whatever means, the sensation is always the same; thus, excitation of the *red*, *green*, or *violet* nerves gives sensations of red, green, or violet respectively, however the excitation be produced,

The ordinates of the three curves in Fig. 146 represent the relative degrees of sensitiveness of the three sets of nerves to the various wave-lengths of light. These curves are from measurements made by Koenig. Thus every wave-length in the spectrum is capable of affecting the red nerves and in some degree producing the sensation of red. The green nerves are more or less affected by all wave-lengths between the Fraunhofer lines *C* and *G*, and the violet nerves are more or less affected by all wave-lengths between the Fraunhofer lines *E* and *H*. Furthermore, a given wave-length or mixture of wave-lengths generally affects all three sets of nerves, giving a blended sensation of color.

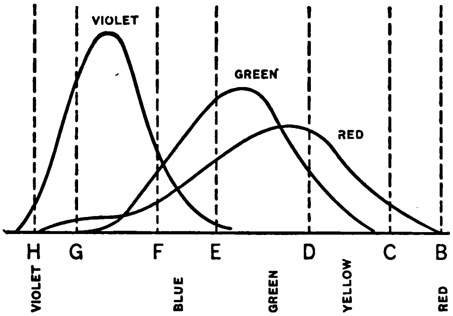

Fig. 146.

138. Contrast effects. — Two complementary colors viewed side by side tend to intensify each other mutually as the eye wanders from the one to the other. Any two colors viewed side by side, if they are at all different, tend to become complementary in appearance as the eye wanders from one to the other. This contrast effect, as it is called, may be shown very strikingly by taking two small pieces of green paper exactly alike and placing one on a large sheet of red paper and the other on a large sheet of blue paper. The two pieces of green paper are so

greatly changed by the opposite contrasts that one can scarcely believe that they are physically alike. A bit of white or grayish paper placed upon a broad sheet of brilliantly colored paper assumes very distinctly a hue complementary to the surrounding color.

The explanation of contrast effects in terms of the Young-Helmholtz theory of color is as follows: White light affects all three sets of nerves approximately equally. A colored light affects all three sets of nerves and the vividness of the color sensation produced depends upon the preponderating excitation of one or two sets of nerves. When two colored lights *A* and *B* are complementary then those nerves which are most affected by *A* are least affected by *B*. When one looks intently at a brilliantly colored light the one or two sets of nerves which are most strongly excited become fatigued. If one then looks at another light these fatigued nerves are less excited than they would be under normal conditions and the second colored light tends to assume a hue which is complementary to the first.

139. Dichroic vision.—Color-blind persons show marked peculiarities in their sensations of brightness and of color.

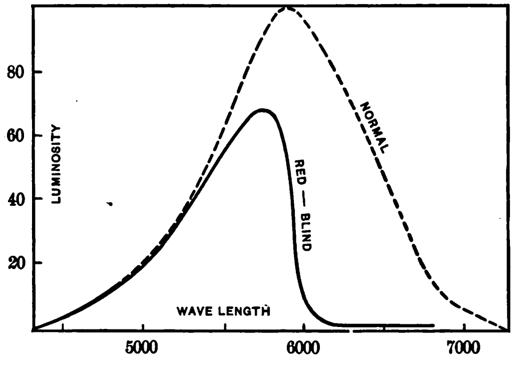

Fig. 147.

Brightness of homogeneous light to color-blind persons.—The ordinates to the dotted curve in Fig. 147 represent the brightness of the various parts of the spectrum of gaslight to a person

who has normal trichroic vision and the ordinates of the dotted curve* represent the brightness of the various parts of the same spectrum to a person who is red-blind. No data are available concerning the brightness of the various parts of the spectrum to a person who is green-blind.

Color sensations due to homogeneous light.—Fig. 148 shows roughly the appearance of the spectrum to persons with normal trichroic vision, to green-blind persons and to red-blind persons. The red and violet nerves of the green-blind person are both affected more or less by any wave-length or mixture of wave-lengths and that particular wave-length in the spectrum that affects these two nerves equally, as ordinary white light affects them equally, gives the sensation called white by the green-blind person. This wave-length lies between the E and F lines according to Fig. 146.

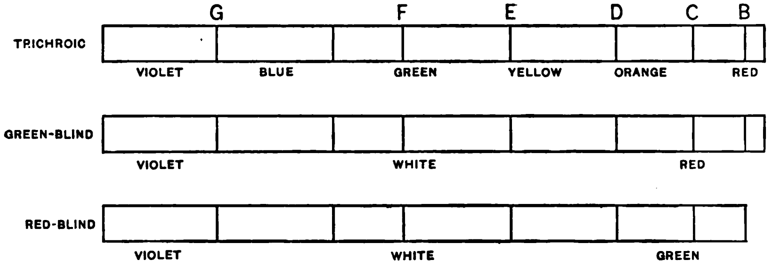

Fig. 148.

The green and violet nerves of the red-blind person are both affected more or less by any wave-length or mixture of wave-lengths, and that particular wave-length (also between the E and F lines according to Fig. 146) which affects these two nerves equally gives the sensation called white by the red-blind person. The red end of the spectrum does not extend much beyond the C line for the red-blind person.

Color sensations due to mixed light.—Color-blind persons show marked peculiarities of color sensation due to mixed

lights. These peculiarities are, of course, very complicated; in fact they are hardly the same for any two persons. The following article which describes the Holmgren test for color-blindness gives some idea of the peculiar color sensations of the color-blind persons produced by mixed light.

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* From measurements by Ferry, *American Journal of Science*, Vol. 44 (1892).

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Red-blind persons invariably select violets and blues as corresponding most closely with the magenta sample.

Both red-blind and green-blind persons are inclined to select brilliant greens as well as brilliant reds in matching the bright red confusion sample.

The selections made by color-blind persons, when attempting to follow their own judgment in matching these confusion sam-

ples in the Holmgren test, are very striking to an observer with normal vision. A university student, who was red-blind, but who showed great power of discrimination in his choice of colors, as viewed from his own standard, and who was fully aware of the nature of his color sense, made the following selections of the worsteds which seemed to him most nearly related to the confusion samples.

Of the twelve skeins of worsted selected to go with the pale green sample, only one contained any green pigment. The remainder were very pale yellows and browns, almost without coloring matter, and two skeins of very pale rose color.

All the worsteds selected on account of their resemblance to the magenta sample were purples in which blue predominated, or nearly pure blues.

With the bright red sample were placed a few other nearly pure reds, together with a much larger number of dark browns and greens.

Surprising as these selections seem to one accustomed to the trichroic color system, they are all readily accounted for under the Young-Helmholtz theory of colors.

CHAPTER XI.

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These special types of transverse waves are most clearly understood by a simple study of the transverse waves produced on a stretched rubber tube one end of which is held in the hand. If the hand is moved rapidly up and down a train of plane polarized waves is produced, the oscillations all along the rubber tube being in a vertical plane. If the hand is moved rapidly in a small circular path, around the stretched tube as an axis, a train of circularly polarized waves is produced and any given particle of the tube describes a circular path as the wave-train passes by it. If the hand is moved rapidly in a small elliptical path a train of elliptically polarized waves is produced, and any given particle of the tube describes an elliptical path as the train of waves passes by it.

Ordinary light. — When one end of the stretched rubber tube used in the above illustration is moved up and down, then to and fro sidewise, then in a circular or elliptical path in irregular succession an irregular series of waves will pass along the tube. Ordinary light, such as comes from a luminous body, consists of an irregular series of waves very similar to these irregular waves

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of the plates are parallel, and is of the plates are at right angles. d by the first plate is *polarized.* wly turned about an axis parallel ity of slowly en the rallel, to xes are the shading in Fig. 149. The fact shows that light waves are trans-

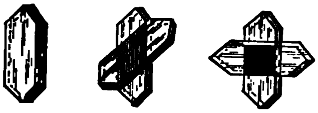
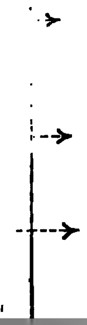


Fig. 149.

light, of plane polarized light, of circu- of elliptically polarized light in terms of — Imagine a large number of thin flat side as shown in Fig. 150, the spaces filled with an elastic jelly. If the board up and down a *plane polarized* train of plane * waves will be transmitted to the right as indicated by the dotted arrows, and each board will oscillate up and down as the wave-train passes by it.



If the board *AB* is moved so that each point in the board *AB* describes a small circular or elliptical path in the plane of the board a *circularly* or *elliptically polarized* train of plane waves will be transmitted to the right as indicated by the dotted arrows, and each board will oscillate in a manner llations of *AB* as the wave-train passes by. tem of boards [illegible] nts the motion of beam of para [illegible] (plane waves) of plane polarized, circularly polarized, and elliptically polarized light respectively. In a beam of ordinary light, these various types of oscillation follow each other with utmost irregularity, any given type of oscillation persisting only for an extremely short interval of time, not longer perhaps than the thousand-millionth part of a second.

A beam of light is said to be *partially polarized* when oscillations in a certain direction preponderate. Such a beam of light varies in brightness when viewed through a slowly rotating tourmaline plate, but is not entirely extinguished by the plate in any position.

143. Polarization by reflection. — Light which is reflected from a polished non-metallic surface is polarized. In general a reflected beam of light is only partially polarized. The degree of polarization varies with the angle of incidence. At normal incidence the reflected beam is not at all polarized. As the incidence becomes more and more oblique the degree of polarization increases, and at certain obliquity of incidence the polarization is complete.* As the incidence becomes still more oblique the degree of polarization again falls off. When the reflected beam is completely polarized it is at right angles to the refracted beam, as shown in Fig. 151.

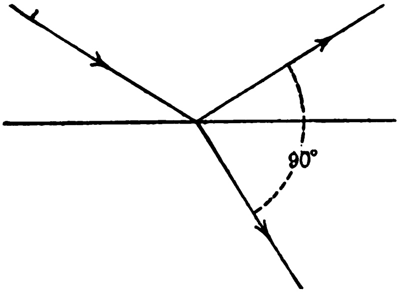


Fig. 151.

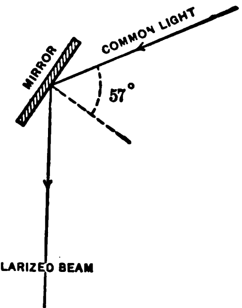


Fig. 152.

The angle of incidence for which the polarization of the reflected beam is complete, is called the *polarizing angle.* Its tangent is equal to the refractive index of the reflecting substance. For glass the polarizing angle is about 57° ; for pure water it is 53° 11′. Fig. 152 shows a glass plate arranged to give a beam of completely polarized light by reflection.

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* If the surface is perfectly clean and perfectly polished.

The oscillations of a beam of light which has been polarized by reflection are parallel to the reflecting surface. Thus the oscillations of the polarized beam in Fig. 152 are perpendicular to the plane of the paper.*

144. Reflection of polarized light from a polished surface.— Consider a beam of polarized light incident at the polarizing angle upon a glass plate. If the glass plate is turned *about the incident beam as an axis*, keeping the angle of incidence constant, the amount of light which is reflected varies from a maximum, when the direction of oscillation of the incident beam is parallel to the surface of the glass, to zero when the glass plate is turned one quarter of a revolution from this position. Fig. 153 shows

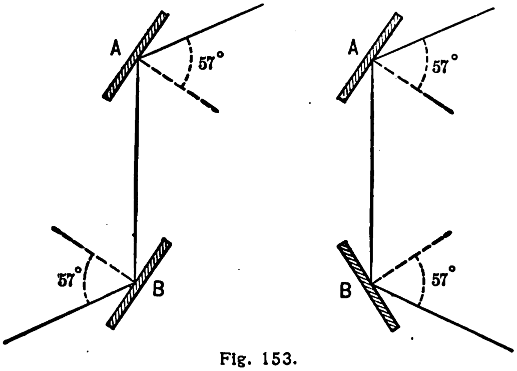

Fig. 153.

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* It may be asked upon what evidence this statement is based ; but it must be remembered that the notions of the elastic-solid theory of light are really unsound and that they are used in this chapter only for the sake of their concreteness. One of the most bitterly contested points in the elastic-solid theory refers precisely to this matter of direction of oscillation, and the bitterness of the contest was due to the entire lack of experimental evidence one way or the other. The whole matter is satisfactorily cleared up in the electromagnetic theory, into a discussion of which we cannot enter here.

the two positions of a pair of glass plates, *A* and *B*, for which the polarized beam reflected from the one is most completely reflected by the other. If either plate is turned one quarter of a revolution about the vertical line *AB*, then no portion of the polarized beam is reflected from the plate *B*.

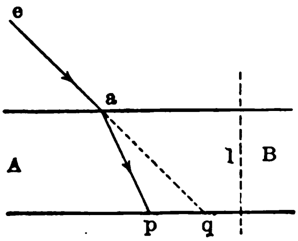

Fig. 154.

145. Double refraction. — Many crystalline substances divide a beam of homogeneous light into two beams by refraction. This phenomenon is called *double refraction*. The crystalline mineral Iceland spar (calcium carbonate) separates the two refracted beams widely and therefore shows the effect very distinctly.

In order to get a clear idea of the phenomenon of double refraction let us first consider a ray of light *e*, Fig. 154, which falls upon the glass plate *AB* and is refracted to the point *p*. If *p* were a luminous point then *pae* would be a ray passing out from *p* and an eye at *e* would see the point *p* at *q*. If the plate were turned about the line *l* as an axis, while the point *p* is stationary, then the point *q* would remain stationary.

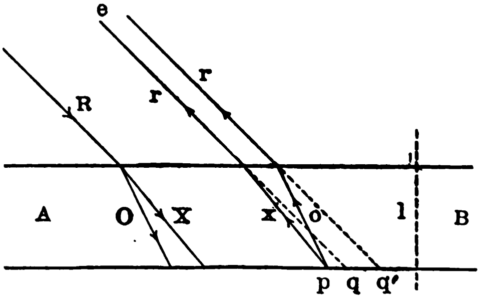

Fig. 155.

A beam of light *R*, Fig. 155, falling upon a plate *AB* of Iceland spar becomes two beams *O* and *X* in the crystal. Conversely a luminous point *p*, Fig. 155, would send out two particular rays *o* and *x*, parallel respectively to *O* and *X*, which would become parallel rays r and r_1 in the air, so that an eye at *e*, Fig. 155, would see *two* images of the point *p* at *q* and *q'* respectively. If the plate of spar *AB* were turned about the line *l* as an axis, while the point *p* remains stationary, then one of the images *q'* would remain stationary just as if *AB* were a plate of glass, and the other image *q* would move round *q'* in a small circular path. The ray *o* or *O* in the crystal which corresponds to the stationary image *q'* is called the *ordinary ray* inasmuch as it is refracted in the ordinary way as in glass; and the ray *x* or *X* in the crystal which corresponds to the moving image *q* is called the *extra-ordinary ray* inasmuch as it is not refracted in the ordinary way. Some crystals (biaxial crystals) divide a beam of common light into two beams, neither of which follows the ordinary laws of refraction.

The rays *r* and *r'*, Fig. 155, are completely polarized, and their planes of oscillation are at right angles. This may be shown by holding a tourmaline plate (or Nicol prism) before the eye at *e*. As the tourmaline plate is turned, one and then the other of the images *q* and *q'* becomes invisible.

Axis of symmetry of Iceland spar. Optic axis. — Any object may be turned one whole revolution about any axis and be in exactly its initial position so that to the eye it is just the same as if the body had not been turned at all. A line in a crystal about which the crystal may be turned $\frac{1}{3}$, $\frac{2}{3}$ or $\frac{3}{3}$ of a revolution, or $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$ or $\frac{4}{4}$ of a revolution, and yet appear to the eye to be in exactly the same position as at first, is called an *axis of symmetry*. A crystal of Iceland spar has such an axis of symmetry. When a ray which enters Iceland spar is in the direction of the axis of symmetry in the crystal, then only one ray exists in the crystal and we have no double refraction. The axis of symmetry of Iceland spar is called its *optic axis*.

146. Huygens' theory of double refraction. — The phenomena of double refraction in Iceland spar were fully analyzed by

Huygens. He assumed two secondary wavelets to pass out from each point of the surface of a plate of Iceland spar as an incident wave reached that point; one of these wavelets being a sphere and the other an ellipsoid of revolution. The envelope of the spherical wavelets determines the ordinary refracted wave, as explained in Art. 31 and the envelope of the ellipsoidal wavelets determines the extraordinary refracted wave.

The spherical and ellipsoidal wavelets which pass out from a point in the refracting surface of a piece of Iceland spar, when an incident wave reaches that point, are most easily described by imagining a center of disturbance inside of the spar so that the wavelets may be a complete sphere and a complete ellipsoid, respectively. Let *p*, Fig. 156, be a center of disturbance in Iceland spar and *AB* the axis of symmetry of the crystal. The circle represents the spherical wave which passes out from *p* and the dotted ellipse represents the ellipsoidal wave. The complete ellipsoid is generated by the rotation of the dotted ellipse about *AB* as an axis.

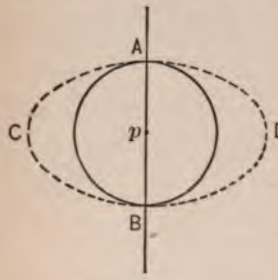

Fig. 156.

Lines drawn outwards from *p*, Fig. 156, are called rays. The disturbance which constitutes the spherical wave and the disturbance which constitutes the ellipsoidal wave may be thought of as traveling outwards along the various rays. The spherical wave is at each point perpendicular to the rays along which it is traveling. The ellipsoidal wave is in general *inclined* to the rays along which it is traveling at each point.

The velocity of the spherical wave is the same in all directions $\frac{1}{1.658}$ as great as the velocity of light in air. The velocity of the ellipsoidal wave is the same in the direction of the axis *AB* as the velocity of the spherical wave. In all directions at right angles to *AB* the velocity of the ellipsoidal wave is $\frac{1}{1.486}$ as great as the velocity of light in air.

optic axis of the crystal. Any plane which is called a *principal plane*.

the spherical wave are everywhere perpen- pal planes, that is, the movement of a spheri- er at the instant that the outwardly moving ches it may be imagined as a momentary the entire shell about *AB* as an axis, followed tch, which brings the shell into its initial

of the ellipsoidal wave are everywhere *in* the hat is the movement of an ellipsoidal shell of the

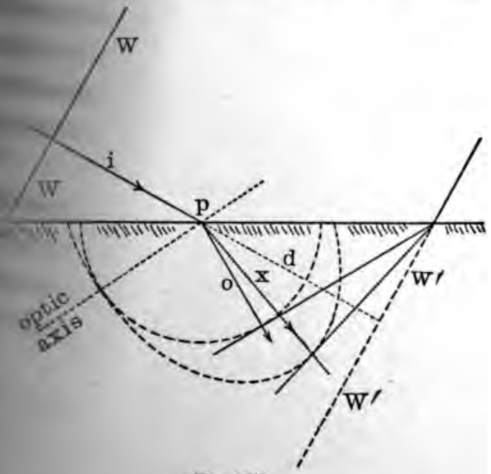

Fig. 157.

the instant that the outwardly moving ellipsoidal wave may be imagined as a momentary twitch which carries of the shell a short distance towards one pole of the ellip- the meridian lines and then back again. The points are the poles of the ellipsoid and the meridian lines are of intersection of the principal planes with the ellipsoid. shows Huygens' construction for the two refracted figure represents the simple case in al lies in the same plane with the incident ray *i* and the ordinary refracted ray *o*.* *WW* represents an advancing wave in air and *W′W′* is the position which this wave would have reached at a given instant if it had not encountered the crystal. It is required to find the positions of the two refracted waves at the given instant. When the wave *WW* reached the point *p* that point was a center of disturbance and at the given instant the wavelets from *p* have had time to travel the distance *d* in air. Therefore the spherical wavelet in the crystal has a radius equal to $\frac{d}{1.658}$, the ellipsoidal wavelet has a minor axis equal to the radius of the spherical wavelet and a major axis equal to $\frac{d}{1.486}$, and the direction of the optic axis being given both wavelets may be drawn as shown. Wavelets from every other point in the refracting surface may be determined in a similar manner. The ordinary wave is the envelope of the spherical wavelets and the extraordinary wave is the envelope of the ellipsoidal wavelets. The ordinary ray *o* is the line drawn from *p* to the point where the ordinary wave touches the spherical wavelet from *p*. This ray is at right angles to the ordinary wave. The extra-ordinary ray *x* is the line drawn from *p* to the point where the extra-ordinary wave touches the ellipsoidal wavelet from *p*. This ray is generally not at right angles to the extra-ordinary wave. The direction of oscillation of the ordinary wave is determined by the direction of oscillation of the spherical wavelet at the point where the spherical wavelet touches the ordinary wave. This direction is perpendicular to the plane of the paper in Fig. 157. The direction of oscillation of the extra-ordinary wave is determined by the direction of oscillation of the ellipsoidal wavelet at the point where the ellipsoidal wavelet touches the extra-ordinary wave. This direction is in the plane of the paper in Fig. 157. The directions of oscillation of the ordinary and extra-ordinary waves is most easily specified as follows: That particular principal plane (a plane which includes the optic axis) which is perpendicular to a refracting surface is often spoken of as *the principal plane*. The oscillations of the ordinary wave are

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* In this case the extra-ordinary ray *x* lies in the same plane with *i* and *o* which is not generally the case.

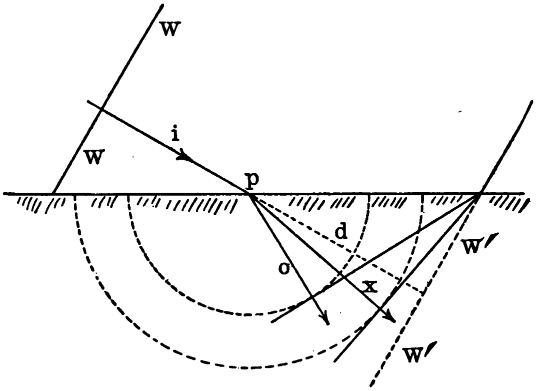

Fig. 158.

at right angles to the principal plane, and the oscillations of the extraordinary wave are in the principal plane.

Fig. 158 shows Huygens' construction for the case in which the optic axis is perpendicular to the plane which includes the incident ray *i* and the ordinary refracted ray *o*. Here the direction of oscillation of the ordinary wave is in the plane of the paper and the direction of oscillation of the extra-ordinary wave is at right angles to the plane of the paper.

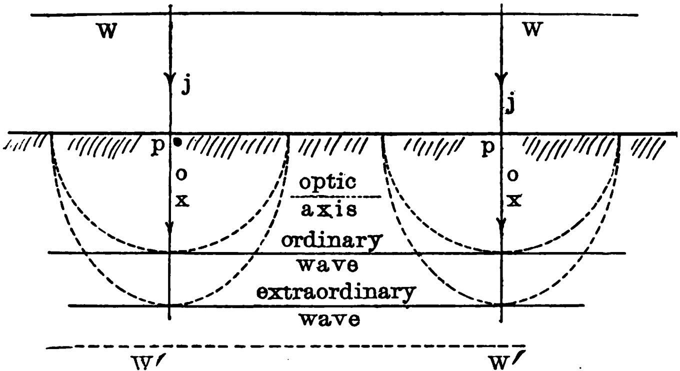

Fig 159.

Fig. 159 shows Huygens' construction for a wave *WW* which is parallel to the surface of the spar, the optic axis being as indicated. Here the ordinary and extra-ordinary rays coincide but there are two distinct refracted waves nevertheless. The directions of oscillation may be easily inferred from principles already laid down.

147. The Nicol prism. — A beam of completely polarized light may be easily obtained by reflection from a glass plate as explained in Art. 143. A more convenient arrangement, however, for obtaining a beam of completely polarized light is the *Nicol prism*, which is constructed as follows: A crystal of Iceland spar is reduced to the form shown in Figs. 160 and 161, the faces of the rhomb being cleavage planes of the crystal. The line *a*, Fig. 161, shows the direction of the optic axis. This rhomb is sawed along the dotted line *AB* perpendicular to the plane of the paper. The faces *AB* and of course the ends of the rhomb are polished and the two pieces are cemented together again with a thin layer of Canada balsam between them.

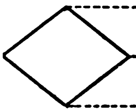

Fig. 160.

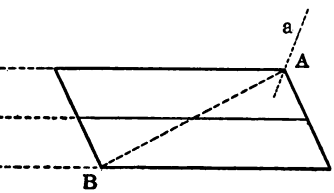

Fig. 161.

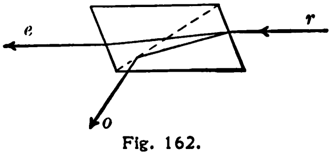

Fig. 162.

A beam of common light *r*, Fig. 162, is broken up by the prism into two beams *o* and *e*. The ordinary beam *o* is *totally reflected* by the layer of balsam while the extaordinary beam passes on alone through the prism as shown. Therefore the light that emerges from the prism is completely polarized and its direction of oscillation is in the plane of the paper in Fig. 162.

The reason for the total reflection of the ordinary ray is as follows: The index of refraction of the spar for the ordinary ray is

1.658 and the index of refraction of the spar for the extraordinary ray is between 1.486 and 1.658, according to its direction, say, 1.550. The index of refraction of the balsam is *between* 1.550 and 1.658. That is, the extraordinary ray goes from a rare to a dense medium in going from spar into balsam, and the ordinary ray would go from a dense to a rare medium in going from spar into balsam. Therefore the ordinary ray may be totally reflected if it strikes the balsam layer at sufficiently oblique incidence, as explained in Art. 47. On the other hand, the extraordinary ray can be only partially reflected however oblique the incidence.

148. Action of the Nicol prism on a beam of polarized light.— Let *ABCD*, Fig. 163, be the front face of a Nicol prism, the same as is shown in Fig. 160.

A beam of polarized light of which the oscillations are parallel to the shorter diagonal *CD*, Fig. 163, becomes wholly the extra-ordinary beam in the prism, since *CD* is the direction of oscillation of the extra-ordinary ray, and the whole of the beam passes on through the prism.

A beam of polarized light of which the oscillations are parallel to the longer diagonal *AB*, Fig. 163, becomes wholly the ordinary beam in the prism, is totally reflected to one side by the layer of balsam, and no portion of it passes through the prism.

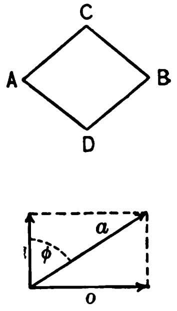

Fig. 163.

Consider a beam of polarized light of which the oscillations are represented in direction and amplitude by the line *a*, Fig. 163. This beam is resolved by the crystalline material of the prism into two beams, ordinary and extra-ordinary, of which the amplitudes of oscillation are represented by the lines *o* and *e*, Fig. 163, respectively. The ordinary ray, only, passes through the prism and its amplitude of oscillation is $a \cos \varphi$. Now, according to Art. 19, the intensity I of the incident beam is to the intensity T of the transmitted beam as a^2 is to $a^2 \cos^2 \varphi$ so that

(22)

The intensity T of the beam which is transmitted by the Nicol prism varies from a maximum, $T = I$, when $\cos^2 \varphi = 1$, to zero when $\cos^2 \varphi = 0$.

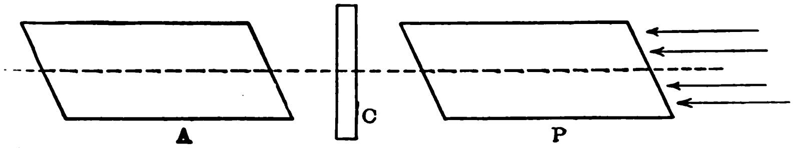

Fig. 164.

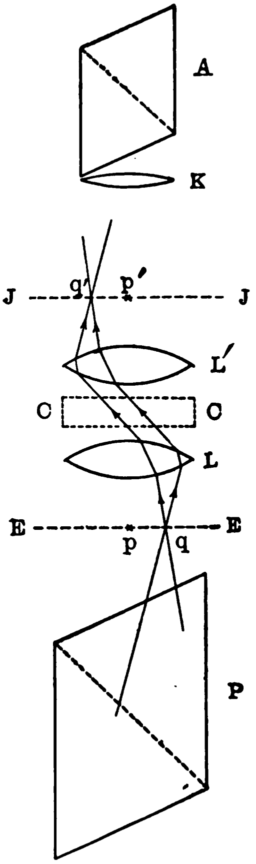

Fig. 165.

149. The polariscope consists of a Nicol prism P, Fig. 164, for producing a beam of polarized light, an arrangement for supporting a transparent object C to be examined, and a second Nicol prism A through which the object C is viewed. The Nicol prism P is called the *polarizer* and the prism A is called the *analyzer.* The analyzer A is mounted so as to turn freely about the axis of the instrument (the dotted line). When the prism A is turned so as to transmit the whole of the polarized beam from P, the prisms are said to be *parallel;* when the prism A is turned so as to cut off all of the polarized beam from P, the prisms are said to be *crossed.*

The arrangement of the polariscope for examining a transparent object in a convergent beam of polarized light is shown in Fig. 165. EE is the principal focal plane of the lens L and JJ is the principal focal plane of the lens L'. Light from any point p or q in the plane EE passes through the object C as a beam of parallel rays and is concentrated at a corresponding point p' or q' in the plane JJ. The light, therefore, which illuminates the central point p' passes through the object in a direction parallel to the axis of the instrument, and the light which illuminates a point q' in the plane JJ passes through the object C in a direction which is the more inclined to the axis of the instrument the farther the point q' is from the central point p'. The lens K is an eye-piece focussed on the plane JJ.

150. Appearance of a plate of a doubly refracting crystal in a polariscope.—The following discussion of the resultant of two oscillations at right angles to each other is a necessary preliminary to the discussion of the action of a crystal plate upon a beam of polarized light. When we speak of the resultant of two oscillations we refer to the movement of a point which performs both oscillations simultaneously. The following discussion refers to Figs. 166 to 174, and to understand the discussion the reader should move a pencil point round and round the circles and ellipses and watch the vertical and horizontal parts of the motion carefully.

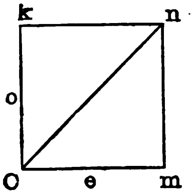

Fig. 166.

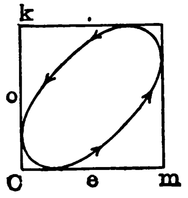

Fig. 167.

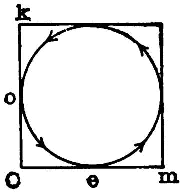

Fig. 168.

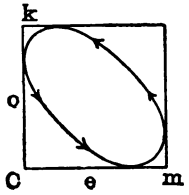

Fig. 169.

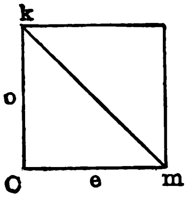

Fig. 170.

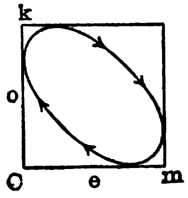

Fig. 171.

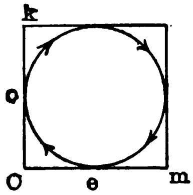

Fig. 172.

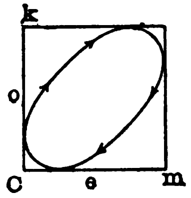

Fig. 173.

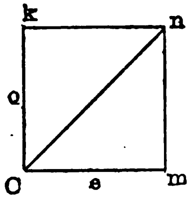

Fig. 174.

Consider an oscillation back and forth along the line *e*, Fig. 166, and another oscillation of the same frequency along the line *o*. If the point which oscillates along *e* is at its turning point m at the same instant that the point which oscillates along o is at its turning point k, then the two oscillations are said to be in phase with each other, and the resultant of the two oscillations is an oscillation along the line Cn.

If the point which oscillates along e has passed its turning point m and is, say, quarter way back towards C when the point o has reached its turning point k, then the oscillation e is about an eighth of a period ahead of the oscillation o in phase, and the resultant of the two oscillations is an elliptical oscillation as shown in Fig. 167.

If the point which oscillates along e has passed its turning point m and is, say, half way back towards C when the point o has reached its turning point k, then the oscillation e is a quarter of a period ahead of the oscillation o in phase, and the resultant of the two oscillations is a circular oscillation as shown in Fig. 168.

When the oscillation e is three eighths of a period ahead of the oscillation o in phase their resultant is the elliptical oscillation shown in Fig. 169. When the oscillation e is half a period ahead of the oscillation o in phase their resultant is the linear oscillation km shown in Fig. 170. The remaining figures 171, 172, 173 and 174 show the resultant of e and o when e is respectively $\frac{5}{8}$, $\frac{6}{8}$, $\frac{7}{8}$ and $\frac{8}{8}$ of a period ahead of o in phase.

Consider a train of plane-polarized plane waves, TT, Fig. 175, of wave-length λ, approaching a crystal plate, CC, as shown. Fig. 176 is a top view of the plate CC of Fig. 175. Let a, Fig. 176, represent the direction and amplitude of the oscillations of the incident light TT. The incident beam is resolved by the crystal plate into two beams of which the oscillations are represented by e and o, Fig. 176. These two beams travel through the crystal at different velocities and one beam, say o, will fall behind the other. Let δ be the distance that beam o has fallen behind e in passing through the crystal plate; this distance δ is called the retardation of beam o. When the two beams of light emerge from the crystal plate, the oscillations of beam o will be

δ/λ periods behind the oscillations of beam e in phase and the resultant oscillation of the emergent beam of light will be similar to some one of the resultant oscillations shown in Figs. 166 to 174, according to the value of δ/λ. Thus, if δ/λ is equal to $\frac{1}{4}$, or to any whole number plus $\frac{1}{4}$, the emergent beam will be a right-handed circularly polarized beam, of which the resultant oscillation is shown in Fig. 168. If δ/λ is equal to $\frac{1}{2}$, or to any whole number plus $\frac{1}{2}$, the emergent beam will be plane-polarized and its direction of oscillation will be *km*, Fig. 170. If δ/λ is equal to $\frac{3}{4}$, or to any whole number plus $\frac{3}{4}$, the emergent beam will be a left-handed circularly polarized beam, of which the resultant oscillation is shown in Fig. 172. If δ/λ is equal to zero or to any whole number, the emergent beam will be like the incident beam in every respect, as shown in Figs. 166 and 174.

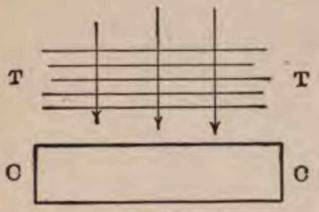

Fig. 175.

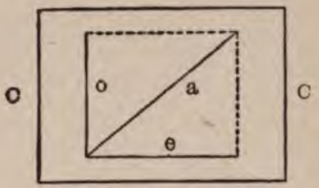

Fig. 176.

These remarkable differences in the character of the emergent beam cannot be detected by the unaided eye, but if the crystal plate is put into the polariscope and viewed through the analyzing Nicol prism the following effects are observed:

(*a*) *Polarizer and analyzer parallel.*—With this arrangement the original polarized beam would be transmitted by the analyzer. The beam after passing through the crystal plate is transmitted by the analyzer if δ/λ is equal to zero or to any whole number; but it is not transmitted by the analyzer if δ/λ is an odd number of halves.

(*b*) *Polarizer and analyzer crossed.*—With this arrangement the original polarized beam would not be transmitted by the analyzer. The beam after passing through the crystal plate is not transmitted by the analyzer if δ/λ is equal to zero or to any whole number; but it is transmitted by the analyzer if δ/λ is an odd number of halves.

If the incident beam of polarized light *TT*, Fig. 175, is white light, containing all wave-lengths, then, with polarizer and analyzer parallel, all wave-lengths are transmitted by the analyzer for which δ/λ is a whole number and all wave-lengths are cut off by the analyzer for which δ/λ is an odd number of halves. When the polarizer and analyzer are crossed then all wave-lengths are cut off by the analyzer for which δ/λ is a whole number and all wave-lengths are transmitted by the analyzer for which δ/λ is an odd number of halves.

If the crystal plate is thick and if the difference in the velocities of the ordinary and extra-ordinary waves is considerable then the retardation δ will be considerable and a great many different wave-lengths between red and violet will satisfy the above condition of transmission by the analyzer so that no color will be visible to the eye. If, however, the crystal plate is thin so that the distance δ is small then only a few wave-lengths, or perhaps but one wave-length, between red and violet will satisfy the above condition of transmission, and brilliant color is produced.

Remark. — If the oscillations of the incident light are parallel to *e* or to *o*, Fig. 176, then the light will pass through the crystal plate as extra-ordinary waves only or as ordinary waves only. Upon emergence the light will be the same in every respect as upon incidence, and the light will be wholly transmitted or wholly cut off by the analyzer according as the prisms of the polariscope are parallel or crossed.

151. Appearance of a plate of a doubly refracting crystal in a polariscope with a convergent beam of polarized light. — The following discussion applies to a plate of Iceland spar cut so that the optic axis is perpendicular to the two faces of the plate.* Let *CC*, Fig. 165, represent the crystal plate in the polariscope and let *CC*, Fig. 177, represent an enlarged view of the plate. The light which illuminates a given point q' of the plane *JJ*, Fig. 165, enters the crystal plate as a parallel beam of plane polarized light more or less inclined to the axis as explained in Art. 149. This beam is resolved by the crystal plate into ordinary and extra-ordinary parts with their directions of oscillation at right angles to each other. The ordinary beam is more or less retarded by the crystal plate relatively to the extra-ordinary beam. Let δ be this relative retardation, then upon emergence the oscillations of the ordinary beam will be δ/λ periods behind the extra-ordinary beam in phase and the resultant oscillation will be like one or the other of Figs. 166 to 174, as explained in Art. 150.

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* For a complete discussion of the optical properties of crystals see Dana, "A Textbook of Mineralogy." Preston's "Theory of Light." Groth's "Physikalische Krystallographie." Liebisch, "Grundriss der Physikalischen Krystallographie."

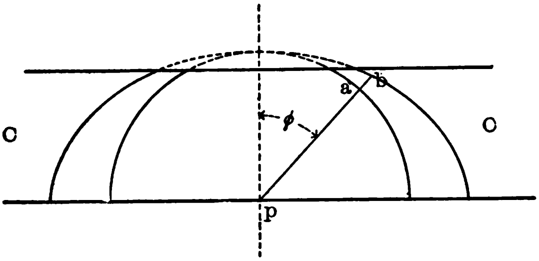

Fig. 177.

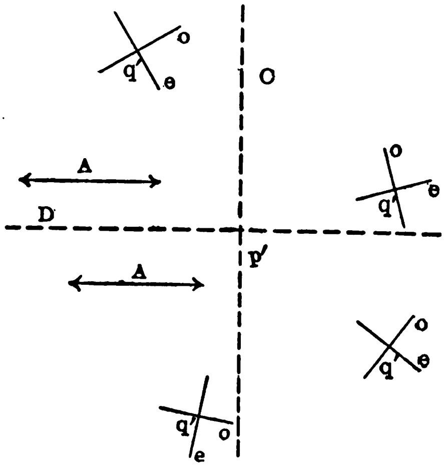

Fig. 178.

Suppose the analyzer to be in the *crossed* position, then those wave-lengths will not be transmitted to the observer's eye for which δ/λ is zero or a whole number.

Consider a ray *pab*, Fig. 177, passing through the crystal plate making an angle ϕ with the optic axis. The retardation δ of the ordinary ray *pa* relative to the extra-ordinary ray *pb* is equal to *ab*, and this retardation increases as ϕ increases. There-

CHAPTER XII.

RADIATION.

154. Radiant heat. — The various homogeneous components (simple wave-trains) of the radiation from a hot or a luminous body have this common property: they generate heat in a body which absorbs them. Such radiation is therefore called *radiant heat.*

The intensity of radiation is properly expressed by its thermal value. Thus the intensity of a beam of light is equal to the number of thermal units (ergs of energy) delivered in one second by the beam, divided by the sectional area of the beam.

155. Luminous effects and chemical effects of radiant heat. — The various homogeneous components of the radiation from a hot body, such as the sun, are physically similar except in wave-length and in intensity. Radiant heat of which the wave-length lies between 39×10^{-6} cm. and 75×10^{-6} cm. affects the optic nerves and produces sensations of light. These limits are not sharply defined, but vary greatly with different persons, with the intensity of the radiation, and with the degree of fatigue of the optic nerves.

The chemical effect of radiation is exemplified in the reduction of carbon dioxide in the growth of plants, in the bleaching action of bright sunlight, in the action of light upon photographic sensitive plates, and so on. The intensity of this chemical action varies greatly with the wave-length. The particular wave-length for which the chemical action is greatest (for a given energy intensity of the radiation) varies with the substance upon which the action is exerted. For the salts of silver used in photography the green, blue, and violet rays are most active, while the extreme red rays are almost wholly inactive.

156. Radiation in a closed region in thermal equilibrium. Normal Radiation. — Ordinarily we think of heat as being radiated only from a body that is hotter than its surroundings. The molecular commotion in the body produces commotion in the ether and the emission of ether waves is the result. Consider a number of bodies in an enclosed space, a room for example. These bodies exchange heat by radiation and they settle to a uniform temperature, thus reaching a state of thermal equilibrium. It is reasonable to suppose that the molecular commotion which still exists in the various bodies when they have settled to thermal equilibrium should continue to produce a commotion of the surrounding ether and that the bodies should continue to emit ether waves. This in fact is the case; the bodies continue to exchange heat by radiation even after having settled to thermal equilibrium but it is a balanced exchange; each body receives heat as fast as it gives off heat. This fact is sometimes referred to as *Prevosts' principle of exchanges*.

A body at a given temperature has a certain definite kind of molecular commotion; that is to say, at a given temperature a certain frequency, f, of molecular vibration is most energetic and other frequencies are less and less energetic the more remote they are from the preponderating frequency f. Furthermore, the ether near a body in thermal equilibrium settles to a state of motion which is entirely conformable to the state of molecular commotion of the body; that is to say, in the ether commotion at a given temperature a certain frequency of oscillation is most energetic and other frequencies of oscillation are less and less energetic the more remote they are from the preponderating frequency. In other words the radiation (ether waves) which exists in a region in thermal equilibrium at a given temperature is an aggregate of wave-trains in which wave-trains of a certain wave-length are most energetic, while wave-trains of other wave-lengths are less and less energetic the more their wave-lengths deviate from the preponderating wave-length.

Remark. — When we wish to refer to the relative intensities of the wave-trains of various wave-lengths which make up a given aggregate of radiation we shall speak of the *composition* of the radiation.

Bodies of the utmost diversity of character may exist together in a region in thermal equilibrium and the ether commotion which conforms to the molecular commotion in one body conforms to that in all. Therefore the radiation in a region in thermal equilibrium at a given temperature is of a definite composition irrespective of the physical characteristics of the body or bodies from which that radiation comes. This radiation is called the *normal radiation* for the given temperature.

Remark. — Two aggregates of radiation are said to be *complementary* to each other when if taken together they constitute normal radiation for a given temperature.

Remark. — The radiation coming *from* a body in a region in thermal equilibrium is not only equal in quantity to the radiation which the body *receives* but it is identical in composition also. The radiation from the body and the radiation to the body are both normal.

Formula for normal radiation. — Let $E \cdot d\lambda$ be the energy between wave-lengths λ and $\lambda + d\lambda$ in the spectrum of a black body, that is in the spectrum of normal radiation, at absolute temperature T. Then

$$E = C\lambda^{-5} \cdot \varepsilon^{-\frac{k}{\lambda T}} \qquad (25)$$

in which ε is the Naperian base and C and k are constants. This formula was deduced by Wien in 1896, and again in a very different and more rigorous manner by Planck in 1900, and it is in very close agreement with experimental results obtained by Lümmer and Pringsheim, by Paschen, and by Paschen and Wanner. See a résumé of recent work on radiation by C. E. Mendenhall and F. A. Saunders, *Astrophysical Journal*, Vol. 13, p. 25, January, 1901. The value of k is 14,500 when λ is expressed in microns (millionths of a meter).

The total energy in that portion of the spectrum of a black body, that is, in the spectrum of normal radiation, which lies between wave-lengths λ' and λ'' is :

$$\begin{aligned} W = {} & C\left(\frac{T}{k}\right)^4 \cdot \lambda'^{-3} \cdot \varepsilon^{-\frac{k}{\lambda' T}} \cdot \left\{\left(\frac{k}{T}\right)^3 + 3\left(\frac{k}{T}\right)^2 \lambda' + 6\left(\frac{k}{T}\right)\lambda'^2 + 6\lambda'^3\right\} \\ & - C\left(\frac{T}{k}\right)^4 \cdot \lambda''^{-3} \cdot \varepsilon^{-\frac{k}{\lambda'' T}} \cdot \left\{\left(\frac{k}{T}\right)^3 + 3\left(\frac{k}{T}\right)^2 \lambda'' + 6\left(\frac{k}{T}\right)\lambda''^2 + 6\lambda''^3\right\} \end{aligned} \tag{26}$$

By putting $\lambda' = 0$ and $\lambda'' = \infty$ this formula gives the total energy radiated (see Stefan's law below); by putting $\lambda' = 0.39$ micron and $\lambda'' = 0.75$ micron this formula gives the total energy in the luminous portion of the spectrum. The ratio of these gives the fractional part of the entire radiation which is included in the luminous portion of the spectrum.

Stefan's law.—The total energy radiated by a black body is proportional to the fourth power of the absolute temperature of the body. This relation was first pointed out by Stefan in 1879. It is involved in the general formula for radiation as explained above.

157. Reflection, Transmission, Absorption, and Emission of Radiation.—When an aggregate of radiation falls upon a body some of the radiation is turned back or *reflected*, some passes on through the body and is said to be *transmitted*, and some is taken up by the body and is said to be *absorbed.*

A primary source of radiation is said to *emit* the radiation it gives off. Thus a lamp is the original or primary source of the light which it gives off and this light is said to be emitted by the lamp.

The radiation coming from a body is in general made up of three parts: (*a*) radiation emitted by the body, (*b*) radiation reflected by the body from other sources, and (*c*) radiation transmitted by the body from sources behind it. In a region in thermal equilibrium the total radiation coming from a body is normal in composition and therefore in a region in thermal equilibrium there is a fixed and simple relationship between emitting power, reflecting power and transmitting power of a substance since emitted radiations, reflected radiations, and transmitted radiations together make up normal radiation for the given temperature. This simple relationship between emission, reflection, and transmission is applied and illustrated in the following articles

It is to be kept in mind that the principle is rigorously applicable only in a region in thermal equilibrium where the radiation received by the body is normal as well as the radiation coming from the body.

158. Proposition. *Emission and absorption are equal.* — Consider a body in a region in thermal equilibrium. The total radiation falling upon the body and the total radiation from it are normal. A portion of the latter is reflected. Imagine this portion removed, and an equal amount subtracted from the incident radiation. Another portion of the total radiation from the body is transmitted; imagine this portion removed and a corresponding amount subtracted from the incident radiation. The remaining portion of the incident radiation is *absorbed* by the body; the remaining portion of the radiation from the body is *emitted*, and these two portions are equal.

159. Selective emission, reflection, and transmission. — In the case of many substances, the total radiation for certain wave-lengths consists almost entirely of *emitted* light, while for other wave-lengths the emission is very slight, and the total radiation is made up chiefly of *transmitted* or *reflected* light. Gases furnish the most striking examples. Such substances are said to exhibit *selective emission.*

When the total radiation for certain wave-lengths contains a greater proportion of transmitted (or reflected) light than is the case for other wave-lengths, the substance is said to exhibit *selective transmission* (or *reflection*).

Remark. — Since emission and absorption are always equal, a body exhibits selective absorption to the same extent that it exhibits selective emission.

160. Ideal cases of selective action. — There are four ideal cases, the consideration of which will help towards the understanding of the behavior of various substances which approximate to them.

(*a*) *Bodies which do not reflect perceptibly.* — Such bodies do not transmit (that is, they do absorb) those wave-lengths which they emit in excess. In this case the transmitted and emitted radiations are complementary.

(*b*) *Bodies which are opaque* (*i. e.*, which do not transmit). — Such bodies emit radiations which are complementary to the radiations which they reflect.

(*c*) *Bodies which do not emit perceptibly.* — Such bodies reflect best those wave-lengths which they do not transmit. That is to say, the reflected radiations and transmitted radiations are complementary.

(*d*) *Opaque bodies which do not reflect* (black bodies). — Such bodies emit normal radiation.

These peculiarities persist to some extent even when the incident radiation is not the normal radiation corresponding to the temperature of the body, although in such a case the two portions into which the radiations are divided are not to be thought of as complementary in the complete sense in which this term is used above.

161. Existing cases of selective action. — (1) Gases do not reflect perceptibly, and they emit those wave-lengths in excess which they do not readily transmit. (See Art. 98.) Ruby glass, which is red by transmitted light, and which shows no marked selective action by reflection, is green when it is heated to incandescence.

(2) Metallic copper is sensibly opaque. It is red by reflected light, but when incandescent, it is *green.*

(3) Fuchsine, of the emitting power of which nothing definite is known, transmits red and violet light and reflects green light almost completely. The approximately complementary character of the transmitted and reflected radiation shows indeed that the emitting power of fuchsine is relatively small.

Gold is yellow by reflected light, and gold leaf is beautifully green by transmitted light. It follows that an extremely thi sheet of gold would not exhibit selective emission when hot. A thick sheet of gold (opaque) appears greenish when incandescent, but the effect is not so marked as in the case of copper.

Remark. — An opaque body which reflects all wave-lengths well emits very little, and an opaque body which reflects very little emits well. Thus dead black steam pipes emit much better than brightly polished ones. Water cools much more slowly in a polished metal vessel than in a blackened one.

162. Black bodies.—An opaque body which reflects very little of the radiation which falls upon it is said to be *black*. Such bodies, when heated to a given temperature, emit very nearly the normal radiation for that temperature. Amorphous carbon, graphitic carbon, and the black oxide of iron are examples of nearly black bodies.

Perfectly black substances do not occur. *A perfectly black body* would be an opaque body which reflected no portion of the rays falling upon it. A small hole in the opaque wall of a large closed chamber is perfectly black, for only an *infinitesimal* portion of the rays which enter find their way out again. If such a chamber be maintained at a uniform temperature, the radiation emitted from the hole will be the same as the radiation within, which is normal.

The appearance of a large, uniformly heated tile kiln, as seen through a hole in the wall, is very striking. Even though the interior be partially free from tiles, nothing can be seen but a flood of soft yellow light. The radiation reflected, transmitted (if any), and emitted by a tile, or by a piece of the air for that matter, is normal, so that identical radiations reach the eye from every portion of the interior. If the peep hole is large enough to cool the adjacent tiles, they become faintly visible; or if a beam of sunlight (it must be light from something hotter than the kiln) is reflected into the opening, the tiles become visible, as if they were in a dark chamber.

163. White bodies. — A body which reflects approximately the same proportion of the various homogeneous components of the radiation which falls upon it is called a *white body*. The light reflected by a white body is of the same composition as the incident light. Thus white paper is red in red light, green in green light, etc. *A perfectly white body* would be one reflecting the whole of the incident radiation, as opposed to a perfectly black body which absorbs the whole. No substance is perfectly white. There is no substance, even, which reflects exactly the same proportion of each homogeneous component of the incident radiation. Most white powders — for example, sugar, magnesium oxide, etc. — reflect the longer wave-lengths in excess, and have a yellowish tint. This is counteracted by the admixture of a small amount of a powder, *e. g.*, ultramarine blue, which reflects the shorter wave-lengths in excess.

Substances, like glass and ice, which transmit (and reflect) every part of the visible spectrum equally well (approximately), are dazzling white when powdered. This is exemplified in the whiteness of snow. Nearly the whole of the light falling upon a sheet of snow is reflected (diffusely), because of the repeated reflections by the successive particles as they are reached by the light as it penetrates deeper and deeper into the snow.

164. Surface color.*— A substance which shows marked selective reflection is said to have *surface color*. All such substances appear different in color by reflected and by transmitted light. Thus gold is yellow by reflected light, and gold-leaf is beautifully green by transmitted light. The aniline dyes, especially when concentrated, show very marked selective reflection, and they are different in color by reflected and by transmitted light.

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* The electro-magnetic theory of light shows: (1) That a perfect electrical conductor would totally reflect all radiations falling upon it; (2) that a substance of which the structural elements have a proper period of undamped electrical vibration would totally reflect wave-trains of that particular period; (3) that in proportion as these proper vibrations are damped the substance would reflect less and absorb more of the trains having its proper period. An example of the first case is furnished by metals, which, especially for the longer wave-lengths, give almost complete reflection. Such reflection is called *metallic reflection*.

Many substances, such as fluorite and fuchsine are examples of the third case, and they reflect certain wave-lengths almost completely. Repeated reflections from fluorite, for example, *isolate* the wave-train which is most readily reflected.

165. Absorption color.—Most colored substances, such as colored glass, colored solutions, etc., show color, perceptibly, by transmitted light only. Many colored substances which appear colored by reflected light, such as the pigments used in painting, really owe their color to selective transmission or absorption. The light falling upon their grains is partially transmitted, becomes colored, and is reflected by numerous foreign particles and by breaks in the continuity of the grains. This is shown by the fact that a mixture of two pigments reflects only those wave-lengths which are reflected by *both* pigments unmixed, just as a pair of colored glasses transmits only those wave-lengths which can pass through both. If the pigments were colored, mainly, by true selective reflection, each isolated grain of each pigment would reflect independently, and all the wave-lengths reflected by each pigment would be found in the light reflected by the mixture.

166. Experimental determination of the composition of radiation. — The radiation of which the composition is to be determined is allowed to pass through the slit of a spectrometer, and the thermal (energy) intensity at each part of the resulting spectrum is observed. This observation is made either by the thermopile, by the bolometer, or by the radiometer.

In the experimental determination of the distribution of energy in the spectrum of a glowing body, the selective transmission of the prisms and lenses of the spectrometer and of the gases and vapors in the atmosphere introduces great errors which cannot be wholly eliminated.

The *thermopile* has been described in Vol. II.

The *bolometer*, invented by Langley * in 1880, consists of a Wheatstone bridge, one of the arms of which is made of a thin strip of blackened metal. This is exposed to the rays, the intensity of which is to be indicated. The radiation causes a rise in temperature and a consequent increase in the resistance of the metal strip. This affects the balance of the bridge, and a sensitive galvanometer in the bridge circuit shows a deflection.

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* See *Transactions of the National Academy of Sciences*, Vol. V.

The *radiometer* used in the measurement of radiation differs essentially from the radiometer of Crookes. It was invented by Ernest Nichols[*] in 1896. It consists of two similar thin vanes of blackened mica, *a* and *a*, Fig. 179, attached to a horizontal arm, and suspended in a high vacuum by means of a fine quartz fiber. The radiation to be indicated falls upon *one* of these vanes and warms it slightly. This causes the few remaining molecules of air to rebound with increased velocity from the blackened face. The reaction pushes the vane backwards, and turns the arm about the fiber as an axis. The deflection is observed by means of a telescope and scale. The sensitiveness of this instrument is such that the rays from a candle at a distance of 450 meters (nearly one-third of a mile) produce a noticeable deflection.

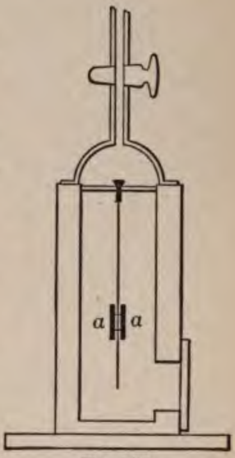

Fig. 179.

Examples. — Langley's determinations of the distribution of energy in the spectra of various sources of radiation afford good illustrations of the use of the bolometer.[†] The ordinates of the curves in Fig. 180 show the relative intensities of the different

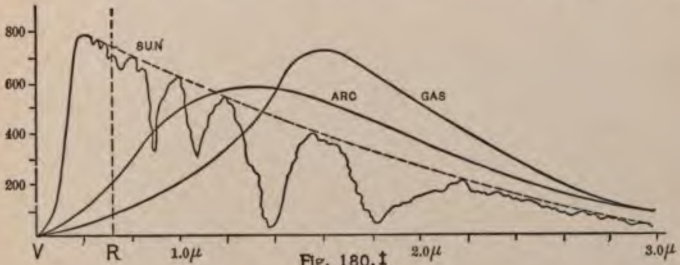

Fig. 180.[‡]

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[*] See *Physical Review*, Vol. IV., p. 297.

[†] *Transactions National Academy of Sciences*, Vol. V.

[‡] The symbol μ in wave-length tables means one micron or a millionth of a meter.

wave-lengths in sunlight, in gaslight, and in the light of the electric arc lamp. The deep notches in the curve of sunlight are absorption bands, due to relatively cool vapors around the sun and in the earth's atmosphere. The dotted line shows approximately what the radiation from the sun would be were it not for this selective absorption.

167. Coefficients of absorption. — Let I be the intensity of a homogeneous beam of light as it enters a plate of indefinite thickness. Let i be the intensity of the beam after having penetrated to a distance x into the plate ; and let Δi be the further decrease in intensity of the beam as it penetrates to an additional distance Δx into the plate. The portion Δi of the beam which is absorbed is, by experiment, sensibly proportional to i and to Δx, so that $\Delta i = -ki \,.\, \Delta x$, or

$$\frac{di}{i} = -k \,.\, dx. \qquad \text{(i)}$$

In this equation k is the proportionality factor. The negative sign is chosen inasmuch as Δi is a decrement of intensity. By integration, equation (i) gives $\log_e i = -kx +$ a constant, or

$$i = Ce^{-kx}, \qquad \text{(ii)}$$

in which C is an undetermined constant. When $x = 0$, then $i = I$. These values substituted in (ii) give $C = I$, so that the equation (ii) becomes

$$i = Ie^{-kx}, \qquad (27)$$

in which I is the intensity of a beam of homogeneous light as it enters an absorbing substance, i is its intensity when it has penetrated to a depth x, e is the Naperian base, and k is a constant called the *coefficient of absorption* of the substance for the particular kind of homogeneous light.

For substances which exhibit selective absorption, the value of k depends upon the wave-length of the homogeneous beam.

The law of decrease of intensity of a beam, as expressed by equation (27), is as follows :

Of the beam which remains unabsorbed the same fractional part is absorbed in each successive layer of the substance.

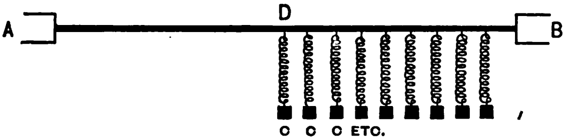

Fig. 181.

168. Helmholtz's theory of dispersion. — Consider a stretched rubber AB, Fig. 181, heavier at one end than at the other, along which transverse waves may be sent by moving the end A up and down. Let c, c, c, etc., be equal weights suspended from the tube by similar spiral springs, so that each of these weights has the same proper period of vibration T. Wave-trains of all wave-lengths will move along AD at the same velocity v, and, ignoring the action of the weights, along DB, the heavier end, at a velocity v/μ. The action of the weights is to cause μ to vary with the periodic time τ of the wave-train. Helmholtz's theory of dispersion is represented mechanically by this arrangement. Helmholtz showed* that the velocity of a wave-train along DB is increased (μ decreased) by the action of the weights, so long as τ is less than T, and decreased (μ increased) so long as τ is greater than T, and that the effect of the weights becomes greater as τ approaches T.

When $\tau = T$, an incident wave-train along AD is almost totally reflected from A, especially if the motion of the weights is frictionless; and, in any case, the action along DB when $\tau = T$ is not of the nature of a wave-train at all. The ordinates of the curve, Fig. 182, represent the values of $\mu \left(\frac{\text{velocity in } DB}{\text{velocity in } AD}\right)$ for different values of the period τ of the incident wave-train. The abscissa of the ordinate T represents the periodic time of the suspended weights.

Examples. — The Helmholtz theory shows that the refractive index of a substance is very greatly increased below an absorption band ($\tau > T$), and greatly decreased above an absorption band ($\tau < T$), and that wave-trains for which $\tau = T$ are almost wholly reflected. This, in fact, is found to be the case for those substances which exhibit powerful selective action and show surface color. This wide variation of the refractive index in the neighborhood of an absorption band has been called *anomalous dispersion.*

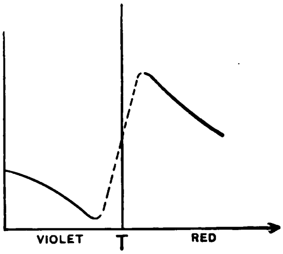

Fig. 182.

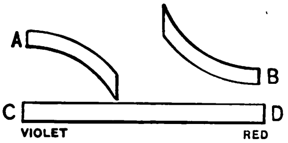

Fig. 183.

169. Example of anomalous dispersion. — The ordinates of the dotted curve in Fig. 182 show the refractive index of a solution of fuchsine for rays of various wave-lengths. The band AB, Fig. 183, shows the actual appearance of a narrow spectrum CD when viewed through a prism containing a solution of fuchsine held in such a way as to deflect the spectrum CD laterally.

170. Phosphorescence and fluorescence. — Some substances, while undergoing slow chemical action, emit radiation of the shorter wave-lengths greatly in excess of the normal amount corresponding to the temperature of the substance. Thus phosphorus oxidizes slowly when it is exposed to the air at a low temperature, and emits a pale white light. This action is sometimes called phosphorescence. The term is, however, more generally applied to the phenomenon described below.

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* See *Berliner Berichte*, 1874, p. 667. Also Winkelmann, "Handbuch der Physik," Vol. II., part I., p. 674.

Many substances — for example, calcium sulphide — glow for a time in the dark, after being exposed to intense radiation. Such substances are said to be *phosphorescent*. The light given off by a phosphorescent substance is generally of greater wave-length than the light which is needed to bring the substance to phosphorescence.

Some substances — as sulphate of quinine, kerosene, and uranium glass — emit light of medium wave-length when exposed to radiation of very short wave-length. Such substances are said to be *fluorescent* or *luminescent*. The Röntgen rays produce strong luminescence in some substances. (See Vol. II.)

SOUND.

CHAPTER XIII.

THE VIBRATION OF A PARTICLE. LOUDNESS, PITCH, AND TIMBRE.

171. Differences between the methods of optics and acoustics. — In the study of light we are concerned chiefly with the phenomena of transmission and the phenomena associated with transmission, such as reflection, refraction, and diffraction. The manner of the generation of light waves and the precise action of light waves upon a body which receives them are matters about which but little is known. In the study of sound, on the other hand, we are concerned chiefly with the manner of the generation of sound waves and with the action that sound waves have upon bodies upon which they fall. The phenomena associated with the transmission of sound such as reflection, refraction, and diffraction are, indeed, as real as in the case of light, but these phenomena are not as a rule of great interest.

172. Simple and compound vibrations. — When a particle moves to and fro along a straight line, performing simple harmonic motion,* its vibrations are called *simple vibrations.* When the to-and-fro motion of a particle is periodic, but not simply harmonic, its vibrations are called *compound vibrations.*

Examples. — The vibrations of a pendulum bob, and the vibrations of the prong of a tuning-fork are simple vibrations. The vibrations of a bar which is slowly lifted and quickly dropped by the successive cogs of a rotating wheel are compound vibrations.

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* Simple harmonic motion is the projection on a fixed straight line of uniform motion in a circle.

Graphical representation of simple and compound vibrations.—Consider a point *p*, Fig. 184, vibrating up and down along the line *AB*. Imagine the paper to move with uniform velocity to the right, then the point *p* will trace a curved line *cc*. If the vibrations of *p* are simple, this curve *cc* will be a *curve of sines*. If the vibrations of *p* are compound, the curve *cc* will be a periodic curve, *i. e.*, each section of it exactly similar to every other section, but not a curve of sines.

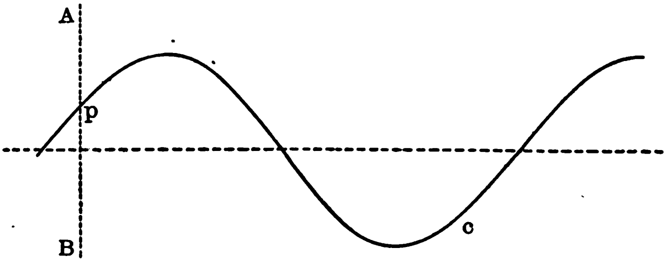

Fig. 184.

The curve shown in Fig. 184 is a curve of sines and it represents a simple vibration. The curve shown in Fig. 185 is periodic but not a curve of sines and it represents a compound vibration. The various curves in Fig. 186 represent actual types of compound vibration which are performed by a point on a violin string.

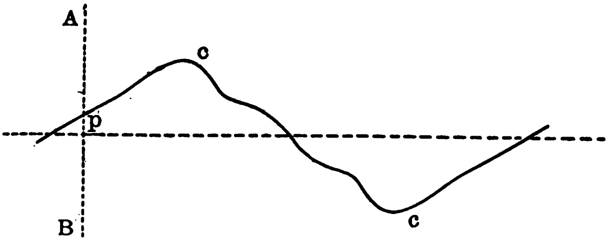

Fig. 185.

Definitions.—The number of (complete) vibrations per second of a vibrating particle is called the *frequency* of the vibrations. The duration of one complete vibration is called the *period* of the vibrations. The whole distance through which the particle moves to and fro is called the *amplitude* of the vibrations.

173. Superposition of simple vibrations. Fourrier's theorem.— A particle may perform simultaneously two or more distinct vibratory movements; in such a case the vibrations are said to be superposed. Thus if the hand be moved slowly up and down and at the same time one finger be moved quickly up and down, the finger will perform both movements, and the moving finger would trace a curve similar to the second or fourth curves in Fig. 186. In this example one vibration is assumed to be of low frequency and the other of high frequency in order that the two movements may not be too greatly confused; as a matter of fact any number of vibratory movements, whatever their amplitudes and frequencies, may be performed by a particle simultaneously.

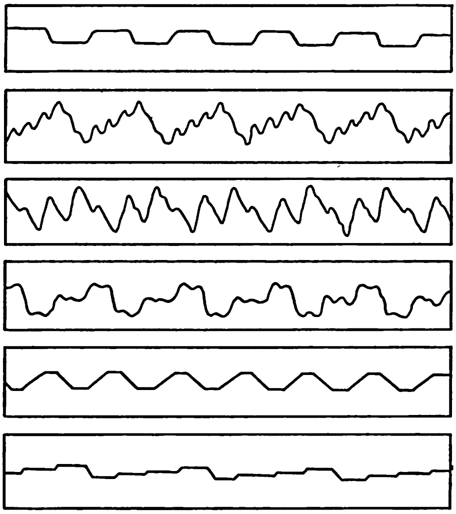

Fig. 186.

Fourrier's theorem.—Any periodic vibration of frequency n, however complicated, may be matched by superposing simple vibrations of which the frequencies are n, $2n$, $3n$, $4n$, and so on, provided the respective amplitudes are properly chosen. That is, any given compound vibration of frequency n may be thought of as composed of a series of superposed simple vibrations of which the frequencies are n, $2n$, $3n$, $4n$, and so on.

Remark 1. — Fourrier's theorem applies to compound wave-trains as well as to compound vibrations. See Art. 17. That is to say, any periodic wave-train may be thought of as composed of a series of simple wave-trains of which the wave-lengths are as the successive fractions 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, etc.

Remark 2. — A sounding body which performs simple vibrations gives off a *simple* train of sound waves. A sounding body which performs compound vibrations gives off a compound train of sound waves. A particle of a medium performs simple vibrations when a simple wave-train passes by it. A particle of a medium performs compound vibrations when a compound wave-train passes by it.

Remark 3. — The physical significance of the distinction between simple and compound vibrations and between simple and compound wave-trains will appear when we take up the study of resonance, or the action of sound waves upon elastic systems such as strings, air columns, bars, and plates.

174. Musical tones. — When sound waves fall upon the ear the result is a sensation of sound. The sensation which is produced by a simple or compound wave-train is called a *tone.* When the wave-train is simple the tone is called a *simple tone.* When the wave-train is compound the tone is called a *compound tone*, or a *clang.*

175. Noises. — Sound sensations not falling under the definition of musical tones, or not consisting of some simple arrangement or combination of musical tones such that the hearer can distinguish the various parts and recognize their relations to one another, may be classified as *noises.*

Rattling noises are due to irregular successions of sharp clicks. *Hissing* and *roaring* noises are due to complex and
ig combinations of tones. In the case of hiss

tones, which may be few in number, are of very high pitch, and in the case of roaring noises the tones are of lower pitch. All manner of combinations of rattling, roaring, and hissing noises occur, from those combinations of musical tones which begin to be so complicated that a hearer cannot distinguish the various parts and recognize their relations to one another, to the extremely complex sounds from waterfalls, trains, and busy streets. The prevalence in a roar of a dominant tone of medium pitch is apt to engage the attention and leave the accompanying noise to a great extent unnoticed.

Musical tones are generally accompanied by characteristic noises. Thus the whispering noises of the breath, and the sounds of the consonants used in articulation, accompany the musical tones of the singer; and the faint noises produced by the fingers, keys, and pedals always accompany piano music. Many noises, on the other hand, are accompanied by characteristic musical tones. A light blow upon a floor, or upon a piece of furniture, produces a faint musical tone, of short duration, which is often prominent enough to be easily distinguishable.

176. Loudness. — That quality of sound which depends upon the *intensity of the sensation* is called *loudness.* The loudness of a given tone, in so far as it is not affected by fatigue of the organs of the ear, and by attention, depends upon the energy of the vibrations which produce it. Tones of medium or high frequency are very much louder, for the same energy of vibration, than tones of low frequency.

The loudness of a tone of given frequency is proportional to the square of the amplitude of the vibrations which produce the sound.

177. Pitch. — That quality of a musical tone which depends upon the frequency of vibrations of the wave-train is called *pitch.* The pitch of a tone is high or low according as the frequency is great or small.

Pitch limits of audibility.— Simple vibrations of lower frequency than about 34 complete vibrations per second are not heard as a musical tone, and when the frequency exceeds 35,000 or 40,000 complete vibrations per second the tone becomes inaudible. These limits vary greatly for different persons.

178. The determination of pitch. — The *direct determination* of pitch is accomplished by virtually counting the number of vibrations in a given time. The *siren* is commonly used for this purpose. It consists of a circular disk mounted on a shaft which is geared to a revolution counter. The disk has one or more circular rows of equidistant holes; it rotates close against the wall of a chamber containing air under pressure, and the holes in the disk come before apertures in this wall in rapid succession. The puffs of air thus produced blend into a musical tone, the pitch of which is known from the observed speed of the disk and number of holes in the row. The disk is sometimes driven by an electric motor and sometimes by the action of the puffs of air. In the latter case the holes in the disk are inclined like the vanes of a windmill.

Another direct method consists in recording the vibrations of the sounding body upon a rapidly moving chronograph cylinder. Sometimes the record is made by means of a stylus attached to the vibrating body and tracing a curve upon a smoked surface. Sometimes a small mirror is attached to the vibrating body and a beam of light is thrown from it upon a photographically sensitized surface. Figure 187 is a facsimile of a tracing obtained in this way. The pitch is determined by counting the number of undulations recorded in a second of time.

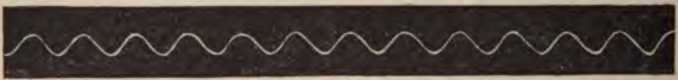

Fig. 187.

Standards of pitch. — A vibrating body which has a constant or nearly constant frequency of vibration may have its pitch accurately determined once for all, and the pitch of any sound may then be determined by comparison with this standard. Standards of pitch are usually in the form of tuning-forks. Tuners of musical instruments always compare the tone to be determined with the tone of the standard fork by the unaided ear. When the difference in pitch is very small this comparison can be made with great accuracy by counting the beats (see Art. 201). When the frequencies of two vibrating bodies are very nearly in a simple ratio such as 1:1, 1:2, 2:3, 3:4, etc., their frequencies may be very accurately compared by a method described in Art. 180.

179. Timbre. — According to Fourrier's theorem a compound vibration of frequency n is composed of superposed simple vibrations of which the respective frequencies are n, $2n$, $3n$, $4n$, etc. Therefore a compound tone or clang of frequency n is composed of a series of simple tones of which the respective frequencies are n, $2n$, $3n$, $4n$, etc. In fact, the simple tones which enter into the composition of a compound tone or clang may be separately distinguished by a practiced ear. This is especially true of the note of a violin, a piano, or an organ; but it is extremely difficult to distinguish the simple tones which enter into the composition of a vowel sound or of a note produced by a singer.

The lowest pitch tone in a clang is called the *fundamental tone* of the clang and the others are called *overtones*. Usually the fundamental tone of a clang greatly preponderates in loudness over the remaining tones. When this is the case the clang has a definite and unmistakable pitch as perceived by the ear.

Two clangs may have the same loudness and the same pitch and still be very different in character. Thus notes of the same pitch and loudness produced by a horn, a clarinet, or a violin are readily distinguishable the one from the others. This variation in the quality of a clang is called *timbre* and it depends upon the relative loudness of the various simple tones of which the clang is composed.

Remark. — The notes of various musical instruments and of the human voice owe their characteristic qualities not only to differences of timbre but to a great extent to the characteristic noises which accompany them and to characteristic ways of beginning and ending.

180. The compounding of simple vibrations which are not in the same direction. Lissajou's figures. — The compounding or superposing of two simple vibrations which are at right angles to each other and of the same frequency is described in Art. 150, and the differences in the resultant vibration due to differences of phase between the two component vibrations are described in detail and shown in Figs. 166 to 174.

Fig. 188 shows the curves traced by a point which vibrates up and down at one frequency, and to right and left at another frequency; these two frequencies being 1 : 1, 1 : 2, 1 : 3, and 2 : 3 respectively. The various shapes of the curves for a given ratio of frequencies are due to differences of phase as described in Art. 150. These curves are called *Lissajou's figures* from their discoverer.

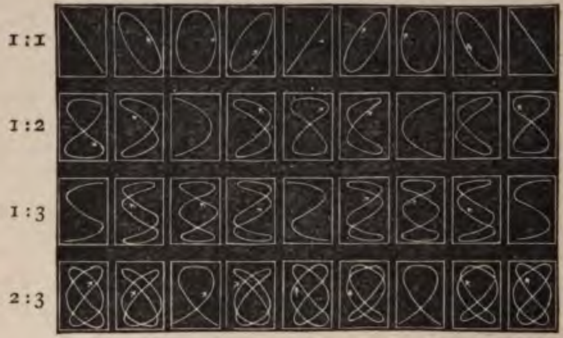

Fig. 188.

Examples. — The simplest method for producing Lissajou's figures is by means of a pendulum which is arranged to vibrate quickly as a short pendulum in the east-west direction, say; and to vibrate slowly as a long pendulum in the north-south direction. The bob of the pendulum may carry a stylus for tracing its motion on a smoked-glass plate.

The string of a guitar may be made to vibrate in a manner somewhat similar to the motion of a jumping rope, that is to vibrate up and down and sidewise simultaneously. These two vibrations are usually not of precisely the same frequency, so that one vibration gains slowly on the other in phase. The result is that a point on the string describes a curve which slowly passes through all the shapes shown in Figs. 166 to 174 (or the first row in Fig. 188).

A microscope * may be mounted on one prong of a tuning fork and caused to vibrate up and down. One may look through this microscope at a sharp point on the prong of another fork which is vibrating from side to side. The point will appear to the eye to partake of both vibratory movements and will describe a Lissajou's curve. The relative frequency of the two forks can be compared with extreme accuracy in this way if their frequencies are nearly in a simple ratio such as 1 : 1, 1 : 2, 2 : 3, etc. This arrangement is the *vibration microscope* of Helmholtz.

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* The object glass only need be mounted on the fork.

CHAPTER XIV.

THE FREE VIBRATIONS OF ELASTIC SYSTEMS.

181. Simple and compound modes of vibration of an elastic system. *Simple modes.* — An elastic system, such as a stretched string or an air column, may vibrate in such a way that every vibrating particle of the system performs *simple harmonic motion* of the same frequency. When a system vibrates in this way there are certain places called *nodes* where there is no motion at all; and the vibrating regions between the nodes are called *vibrating segments*. A system vibrating in this way is said to vibrate in a *simple mode*.

A given elastic system may break up into vibrating segments in a great variety of ways; that is, the system may vibrate in a great variety of simple modes, each mode having a characteristic frequency of its own.

These general statements are beautifully exemplified by a metal plate set vibrating by drawing a violin bow across its edge. Such a plate may be made to give off any one of a great variety of simple tones. To this end the vibrations of the plate must be controlled by placing the finger lightly against the plate at one point or another while drawing the bow. The nodal lines may be shown by strewing fine sand upon the plate. This sand is thrown away from the vibrating segments and it collects along the nodal lines. These sand figures were first studied by Chladni. Fig. 189 shows some of the sand figures obtained by Chladni and depicted in his treatise on Acoustics (Leipzig, 1787).

In a vibrating string *nodal points* separate the vibrating segments; in a vibrating plate *nodal lines* separate the vibrating segments, and in a system of three dimensions *nodal surfaces* separate the vibrating segments.

Compound modes. — An elastic system may perform any number of its simple modes of vibration simultaneously. In this case the system is said to vibrate in a *compound mode.* Each particle of the system performs, in general, compound vibrations, that is to say, each particle performs simultaneously the various simple harmonic movements corresponding to the respective simple modes that are present in the vibratory motion of the system; and, of course, the vibrating system gives off a compound tone of which the simple component tones correspond to the respective simple modes that are present in the vibratory motion of the system. The relative intensity or violence of the various simple modes of a vibrating system depends upon the manner in which the system is set into vibration.

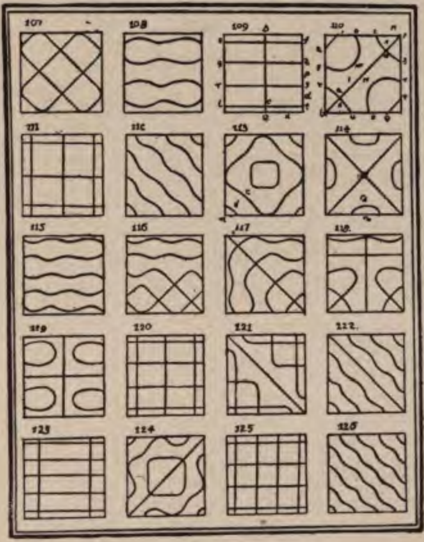

Fig. 189.

Example. — A metal plate struck by a hammer vibrates simultaneously in a great many simple modes and gives off a complex sound composed of the many corresponding simple tones. A little practice enables these various tones to be distinctly heard. The relative intensity of the various simple modes alters greatly with the location of the hammer blow and with the nature of the face of the hammer, that is, according as the hammer face is hard and bare or muffled with soft cloth.

Method of the following discussion. — The vibrations of an elastic system may in every case be looked upon as a *stationary wave-train* due to repeated reflections of an original wave disturbance from the boundaries of the system. The study of vibrating air columns and of vibrating strings, in particular, is greatly simplified by this method of treatment and consequently this method is employed in the following articles.

182. The transverse vibration of a string.* — If a stretched string be struck, or distorted and released, the wave produced will travel at a velocity

$$v = \sqrt{\frac{T}{m}} \qquad (a)$$

in which T is the tension of the string in dynes and m is the mass of the string in grams per centimeter of length. See Appendix A.

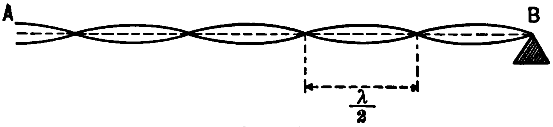

Fig. 190.

Consider an indefinitely long stretched string *AB*, Fig. 190, fixed to a rigid support at *B*. Imagine a simple transverse wave-train of wave-length λ to be approaching from *A* towards *B*. This wave-train will be reflected at *B* (with change of phase) and the reflected train and advancing train will together form a stationary train of which the nodes are distant λ/2 from each other, as shown in Fig. 190.

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* The student should again read Arts. 21 and 22 on stationary wave-trains.

Once this stationary train is established a rigid support might be placed under the string at any one of the nodes and the string between this new support and B would continue its vibratory motion undisturbed and independently of the remainder of the string except, of course, that its motion would be slowly stopped by friction. The length, l, of a vibrating string may therefore be any multiple, n, of $\lambda/2$. That is:

$$l = \frac{n\lambda}{2} \qquad (b)$$

in which n is any whole number. Let τ be the period of one oscillation of the string, then the number of oscillations per second, or frequency, is $f = \frac{1}{\tau}$. Furthermore the relation between λ, v and τ is $\lambda = \tau v$ as explained in Art. 15. Therefore

$$f = \frac{v}{\lambda}. \qquad (c)$$

Substituting the value of v from (a) in (c) and the value of λ from (b) in (c) we have:

$$f = \frac{n}{2l}\sqrt{\frac{T}{m}} \qquad (28)$$

in which f is the frequency of oscillation of a string of length l, stretched with tension T, and weighing m grams per centimeter, and n is any whole number.

When n is unity, the whole string is one vibrating segment. The tone given in this case is called the *fundamental* of the string. When n equals 2, 3, or 4, etc., the string has 2, 3, or 4, etc., vibrating segments. The tone given by the string in this case is called an *overtone* or a *harmonic*.

Simple and compound modes of vibration of a string. — A string vibrating so as to give its fundamental tone or one of its harmonics alone, vibrates in a simple mode. A string may (and generally does) vibrate so as to give simultaneously its fundamental tone together with its various harmonics. In such a case the string vibrates in a compound mode.

The relative intensities of the fundamental and the various harmonics of a vibrating string depend upon the manner of exciting the vibrations. Thus the plucked string of the guitar, the struck string of the piano, and the bowed string of the violin give very different clangs, and these clangs change very perceptibly with the point at which the string is plucked or struck or bowed. A guitar string plucked near the center gives only odd harmonics and those not strongly. If plucked about one seventh of the length of the string from one end, the overtones up to the sixth are prominent, and the clang is correspondingly rich.

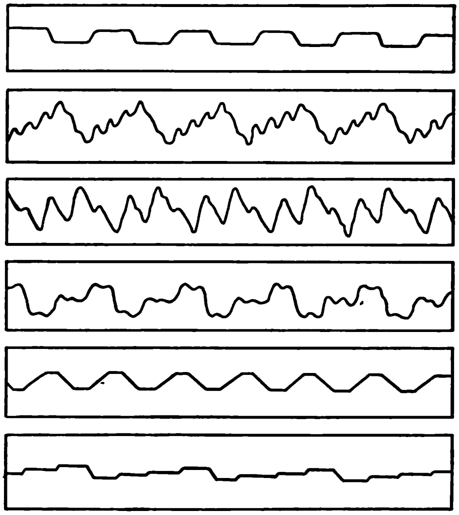

Fig. 191.

A skilled violinist can produce a very great variety of compound vibrations of a string by varying the point of application of the bow, the velocity of the bow, its pressure on the string, etc. The curves in Fig. 191 illustrate a few types of compound vibration of a bowed string. These curves are selected from the numerous tracings published by Krigar-Menzel and Raps.* The curves were obtained by photographing upon a moving plate the up-and-down movements of an isolated point (or element) of the string.

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* *Wiedemann's Annalen*, Vol. 44, p. 623.

183. The vibration of air columns.—The following discussion depends upon the reflection of air waves (sound waves) in a pipe from the closed end of the pipe and from the open end of the pipe. The reflection from a closed end is complete and it takes place with change of phase, inasmuch as the air near the closed end is not free to move and a node must exist there. The reflection from the open end of a pipe is partial. The fractional part of a wave-train that is reflected varies from 80 per cent. to about 97 per cent. according to the relation between the wave-length of the train and the diameter of the pipe. The reflection that does take place is without change of phase, inasmuch as the air at the open end of the pipe has great freedom of motion and an antinode must exist there.

Let *AB* (Fig. 192) be an indefinitely long tube filled with air, and open at the end *B*. Consider a simple train of sound waves of wave-length λ advancing in the tube from *A* towards *B*.

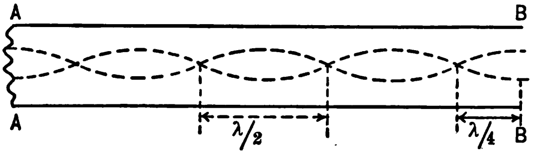

Fig. 192.

When this train reaches *B*, it is almost entirely reflected, without change of phase. The reflected train superposed upon the advancing train produces a stationary train of which the nodes are distant λ/2 from each other, the first node being at a distance of λ/4 from the open end, which is an antinode, as shown by the dotted wave lines in Fig. 192.

The air in each vibrating segment of this stationary wave surges back and forth in the direction of the tube; the air on the two sides of a node surges towards the node simultaneously, and then away from it simultaneously, so that the air at the node is alternately compressed and expanded, but does not move.

The time τ required for one back-and-forth movement, that is for one vibration of the air in the tube, is the period of the wave-train, and from equation (1) of Art. 15 we have

$$\tau = \frac{\lambda}{v} \qquad (a)$$

whence, since the frequency, f, is equal to $1/\tau$, we have

$$f = \frac{v}{\lambda} \qquad (b)$$

in which v is the velocity of sound waves in the tube; this velocity is sensibly the same as the velocity of sound in the open air.

Air column open at both ends. — When once the stationary wave-train (or vibration) is established in the tube AB, the tube may be supposedly cut off and an open end left at *any antinode* without altering * the subsequent action in the detached portion of the tube, except that the vibrations would soon die out as the energy is dissipated. This dissipation is partly due to the friction of the air as it surges back and forth along the walls of the tube and partly due to incomplete reflection from the open ends of the tube.

Suppose a tube AB containing a stationary wave-train of wave-length λ to be cut off at the nth antinode from the open end B. Then the length l of the detached portion will be

$$l = \frac{n\lambda}{2} \qquad (c)$$

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* The cut must be made at an antinode if the end is to be left open and the stationary wave-train unaltered. The original wave-train is reflected from the open end B without change of phase, thus producing an antinode at B. This reflected train is again reflected from the cut-off end without change of phase, thus producing an antinode at the cut-off end. This second reflected train takes the place of the original advancing wave-train in the cut-off portion of the tube.

Now if λ had been only half as great, the same value of l would be obtained by choosing n twice as great; if λ had been only one third as great the same value of l would be obtained by choosing n three times as great, etc. Therefore, a tube of given length l may accommodate a stationary wave-train of any wave-length that will satisfy equation (c), n being any whole number. Conversely a wave-train of given length λ may be accommodated by a tube of any length that will satisfy equation (c), n being any whole number.

Substituting the value of λ from (c) in (b), we have

$$f = n \cdot \frac{v}{2l} \qquad (29)$$

in which f is the frequency of vibration of an air column open at both ends, l is the length of the air column, v is the velocity of sound, and n is any whole number. The frequency corresponding to $n = 1$ is the lowest possible frequency of vibration of the given air column, and the tone produced when $n = 1$ is called the *fundamental* tone of the column. The tones produced by the vibrations of the air column when $n = 2$, 3, etc., are called the second, third, etc., *overtones* or *harmonics* of the air column.

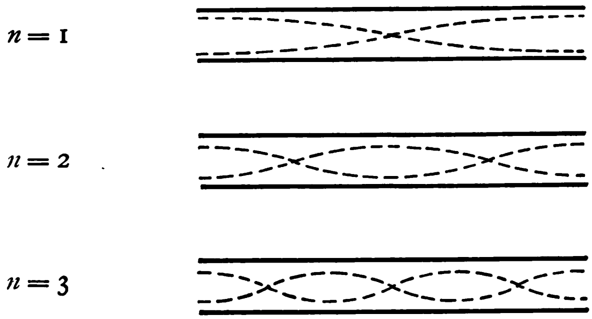

Fig. 193.

The character of the vibration in an air column of a given length open at both ends when $n = 1$, when $n = 2$, and when $n = 3$ is shown by the dotted wave lines in Fig. 193.

Air column closed at one end. — When the stationary wave-train is once established in the tube *AB*, Fig. 192, a rigid diaphragm may be supposedly placed across the tube at any node without altering the subsequent behavior; except that the vibrations will soon die away. In this case, the length l of the detached portion of the tube will be

$$l = \frac{n\lambda}{4}, \tag{d}$$

in which n is an *odd* number. Substituting the value of λ from (d) in (b), we have

$$f = n\frac{v}{4l}, \tag{30}$$

in which f is the frequency of vibration of an air column of length l, closed at one end. Since n in this case is necessarily an odd number, an air column open at one end has only odd harmonics. The character of the vibrations when $n = 1$, when $n = 3$, and when $n = 5$ is shown by the dotted wave lines in Fig. 194.

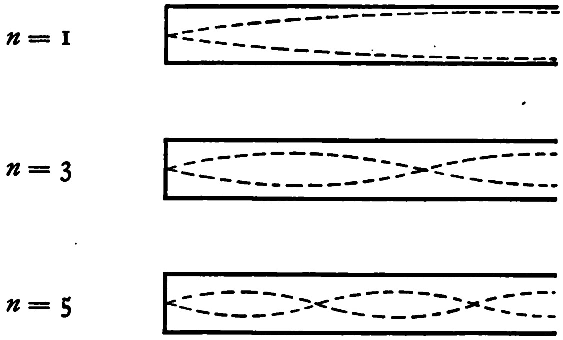

Fig. 194.

Simple and compound modes of vibration of an air column. — An air column vibrating so as to give its fundamental tone or one of its harmonics alone, vibrates in a simple mode. An air column may (and generally does) vibrate so as to give simultaneously its fundamental together with the various harmonics. In such a case the air column vibrates in a compound mode. The relative loudness of the various overtones depends upon the shape of the air column and upon the manner in which the vibrations are excited. This is exemplified by the strikingly different clangs of organ pipes, horns, and clarinets.

184. Organ pipes. — The vibrating elements of the pipe organ are columns of air called *organ pipes*. These are very similar to the bark whistles known to children. A metal or wooden tube *AB*, Fig. 195, incloses the vibrating air column, and the vibrations are excited by an air jet which, issuing from a narrow slit, blows across the opening against the sharp edge *s*. This air jet makes a slight noise, which starts the air column vibrating feebly. These vibrations react upon the air jet and cause it to play from one side to the other of *s*. This reinforces the vibrations, so that they quickly become energetic. When the end *A* is closed, the air column vibrates in accordance with equation (30), and only the odd overtones are present. When the end *A* is open, the air column vibrates in accordance with equation (29), and both even and odd overtones are present. With broad pipes the overtones are very weak, and the tone approaches a pure tone in character. With narrow pipes the overtones up to the fifth or sixth are audible. When an organ pipe is blown strongly, the overtones become more prominent. If blown very strongly a second or third overtone may become so prominent as to completely dominate the tone.

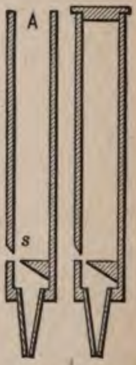

Fig. 195.

The *reed pipe* is an air column into which a stream of air flows through an opening in and out of which a spring or *reed* vibrates, so as to convert the air stream into a series of puffs of the proper rhythm to excite the vibrations. The clarinet, the cornet, and the vocal organs of man are types of the reed pipe.

185. The clarinet. — The tones of the clarinet are produced by the vibrations of an air column, the length of which may be altered at will by uncovering openings in the side of the tube. The end of the tube which is placed in the mouth is covered by a light *reed* or *tongue*. When the player blows into the instrument, this reed, acting like a valve, suddenly closes the opening, and a wave passes down the tube, is reflected from the open end of the tube, and, returning, strikes the reed and causes it to open again. The impulse is then repeated. The movement of the reed is not, however, entirely controlled by the surging of the air in the tube of the instrument. The lips of the player are pressed against the reed in such a way that its tendency is to open and close in the proper rhythm, independently of the surging of the air.

The mouth end of a clarinet is not open by any means nor is it tightly closed, still as a matter of fact the clarinet has only odd harmonics, which shows that the air column is effectually closed at the mouth end of the instrument.

186. The cornet. — In the case of the cornet the lips of the performer take the place of the reed of the clarinet, and the length of the vibrating column of air is altered at will by means of valves or keys which include or exclude auxiliary lengths of tube. The keys provide for six distinct lengths of air column and by sounding the various harmonics of these six lengths at will, the player produces any desired note. The bugle is similar to the cornet except that it has a tube of fixed length, the harmonics of which are sounded at the will of the player. The harmonics ordinarily used are the second, third, fourth, fifth, and sixth.

187. The vocal organs. — These consist of the *vocal cords*, two muscular membranes which are stretched across the top of the windpipe, and the *mouth cavity*. The vocal cords are tuned to any desired pitch by muscular effort and are set vibrating by air forced from the lungs.

The smooth tones produced in singing arise from vibrations in which the vocal cords do not strike against each other. In speech the chords strike against each other as they vibrate and produce sounds which contain a great many simple component tones. The vowel sounds are produced by so shaping the mouth cavity as to strengthen (by resonance) certain of these component tones at will. The consonant sounds, so common in speech, are characteristic noises with which, in articulation, the vowel sounds are begun and ended.

188. The longitudinal vibration of rods and strings. — A brass rod supported at the center and rubbed lengthwise with a rosined cloth may be set into vibration of a kind exactly similar to the vibration of the air in a tube open at both ends. That is, the metal at each point, except of course at a node, vibrates back and forth in the direction of the length of the rod. The metal on the two sides of a node surges towards the node simultaneously, then away from the node simultaneously so that the metal at the node is alternately compressed and stretched. The frequency of vibration of a metal rod vibrating longitudinally is given by equation (29), in which v is the velocity of longitudinal waves along the rod. The dotted wave lines in Fig. 193 show the character of the longitudinal vibration of a rod free at both ends with one, two, and three nodes respectively.

A string stretched between rigid supports vibrates longitudinally in a manner similar to an air column closed at both ends.

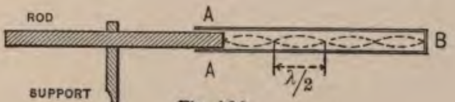

Fig. 196.

189. Kundt's experiment. — A rod (Fig. 196) is supported, say at its center, and set vibrating longitudinally by rubbing it with a rosined cloth. One end of this vibrating rod extends loosely into a tube of air *AB*. A train of waves passes out from the end of the rod as it vibrates back and forth, and this wave train upon reflection from the closed end *B* forms a stationary train in the tube. When the rod is adjusted until its end is near a node, this stationary train acquires great intensity.* Lycopodium or other light powder, which is strewn inside the tube *AB*, is swept out of the vibrating segments by the surging motion of the air and is heaped up at the nodes. The distance between nodes is then easily measured. The distance is half the distance traveled by the sound waves in the air of the tube during one vibration of the rod. If the rod is given its fundamental tone, for which $n = 1$, the length of the rod is half the distance traveled by a sound wave in the rod during one vibration. Therefore the length of the rod divided by the distance between nodes in *AB* is equal to the velocity of sound in the material of the rod divided by its velocity in air. Knowing the velocity of sound in air, its velocity in the rod is thus easily determined. The tube *AB* may then be filled with any gas and the distance between the nodes again measured; whence the velocity of sound in the gas may be found. Some of the velocities given in Art. 11 were determined in this way.

190. Transverse vibrations of stiff rods and plates. — The velocity of propagation of a transverse wave (a bend) along a stiff rod or plate varies in a complicated manner with the wave-length. Therefore the frequencies of the various simple modes of vibration of a stiff rod or of an elastic plate are not simply related to each other as are the frequencies of the various overtones of strings and air columns, and a stiff rod or plate vibrating in a compound mode gives a clang of which the constituent simple tones are *unharmonic.** This is exemplified by the discordant sounds produced by cymbals and gongs.

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* The end of the rod is an antinode of the stationary wave train on the rod but the actual distance through which the end of the rod moves back and forth is very small. If the end of the rod is moved until it is at an antinode of the air wave train in the tube then the motion of the air train at the antinode will be the same as the motion of the end of the rod. As the end of the rod is moved farther and farther away from the antinode of the air train the motion of the air train at the end of the rod continues to conform approximately to the motion of the end of the rod and the motion of the air at the antinode of the air train increases more and more, a maximum violence of the air train being obtained when the end of the rod is very near a node of the air train.

The bell, as a rule, gives an unharmonic series of tones. Bell founders have, however, learned to model bells in such a way as to make the more prominent tones harmonic, thus producing a bell which gives a rich, mellow tone.

Fig. 197 shows the more prominent tones of a Russian bell in the library of Cornell University.

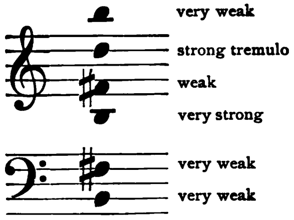

Fig. 197.

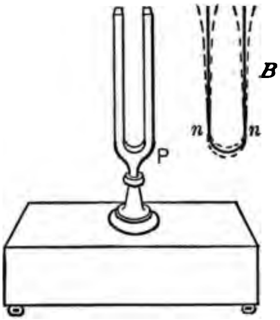

Fig. 198.

191. The tuning fork is a stiff rod bent into the form shown in Fig. 198. The character of its fundamental mode of vibration is shown by the lines *B*. The two nodes *nn* are near together, and the intervening segment, together with the metal post *P*, moves up and down through a small amplitude and causes the sounding board upon which the fork is mounted to vibrate in unison with the fork. The second tone of a fork, which is usually two or three octaves above its fundamental, dies out quickly after the fork is thrown into vibration, leaving the fundamental alone. The tuning fork, therefore, gives a simple tone. Carefully tuned forks are much used by musicians as standards of pitch.

192. Vibrating diaphragms.—The sound produced by a vibrating membrane owes its peculiar quality largely to the characteristic quickness with which the vibrations die out. The sound of the drum, for example, is so brief that there is scarcely time for a distinct sensation of pitch to be produced. The overtones of a vibrating membrane or diaphragm are *unharmonic*.

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* That is, not related in frequencies as the successive whole numbers.

A diaphragm, being light and exposing a broad surface to the air, vibrates very perceptibly with the air when any sound strikes it. It is this property of a light diaphragm which renders it useful in the telephone and the phonograph.

CHAPTER XV.

IMPRESSED VIBRATIONS AND RESONANCE.

193. Free vibrations; impressed vibrations. — The vibrations which a body performs when struck or disturbed in any way, and left to itself, are called its *free* or *proper* vibrations.

When a simple train of sound waves of any wave-length strikes a body, the body is made to vibrate in unison, or in the same rhythm, with the impinging waves. Such vibrations are called *forced* or *impressed* vibrations. A compound wave-train causes a body to perform simultaneously the simple vibrations corresponding to the various simple wave-trains which enter into the composition of the compound train.

The quickness with which a body assumes a steady state of impressed vibration under the action of sound waves depends upon the mass of the body and upon the extent of the surface which it exposes to the action of the waves.

The violence of the impressed vibrations depends upon the intensity of the impinging waves, upon the extent to which the vibrations are damped, and upon the relation between the frequency of the impinging waves and the frequency of the free vibrations of the body.

194. Damping. — Vibrations are said to be *damped* when they die out quickly. This is generally due, in part, to the dissipation of energy in the body in the form of heat, as it is repeatedly distorted, and in part to the giving up of energy to the surrounding air. Thus the vibrations of light bodies, which expose considerable surface to the air (diaphragms, etc.), and of bodies which are imperfectly elastic, are greatly damped. On the other hand, the vibrations of a heavy, elastic body, such as a tuning fork, are but slightly damped. A heavy tuning fork performs several thousands of perceptible vibrations when struck. The column of air in an organ pipe performs several hundreds of perceptible vibrations after the exciting cause ceases. A drum head performs only a very few perceptible vibrations when struck.

195. Resonance. — Very perceptible vibrations are impressed upon diaphragms and stretched membranes by sound wave-trains of any frequency. *The vibrations impressed upon a heavy body, of which the damping is slight, are much more violent when the impressed frequency approaches the proper frequency of the body.* Thus the sound of a tuning fork (removed from its sounding board) is perceptibly enforced when it is held near the open end of a tube of any length. If the length of the air column is adjusted, as, for example, by pouring water into the tube, the sound becomes louder as the proper frequency of the air column approaches that of the fork; and it reaches a very distinct maximum when the impressed vibrations become proper to the air column. With bodies which exhibit less and less damping the maximum violence of impressed vibrations at proper frequency becomes more and more pronounced; and at the same time the impressed vibrations of improper frequency become more and more nearly imperceptible. Thus a massive tuning fork, mounted upon its sounding board, is thrown into quite violent vibration by a tone of its proper frequency sustained for four or five seconds, but it is scarcely affected by a tone of which the frequency differs by one per cent. from the proper frequency of the fork.

This pronounced maximum violence of impressed vibration at proper frequency is called *resonance*, and the vibrating body or air column is called a *resonator*.

When impressed vibrations are proper to a body, the action of the impulse due to each successive wave is to add to the existing motion, and the vibrations increase in violence until the energy given to the vibrating body by the impinging waves is all dissipated by damping. It is for this reason that the impressed vibrations become quite violent when the damping is small. On the other hand, when improper vibrations are impressed upon a body, the periodic forces with which the waves act upon the body must take the place more or less of the *internal elastic* forces which ordinarily cause a body to vibrate. Consequently the vibrations cannot become violent.

Remark. — A resonator of which the proper frequency coincides with the frequency of one of the component tones of a compound tone responds to the component tone by resonance and strengthens it. Thus if the frequencies of the fundamental and successive overtones of a string are 100, 200, 300, 400, etc., vibrations per second, a resonator of which the proper frequency is 400 will be set into more or less violent vibration by the sound of the string, and this particular overtone will be made louder.

196. Analysis of compound tones by means of resonators. —Any component tone of a clang may be easily detected, or brought to notice, by intermittently strengthening it by means of a resonator tuned to unison with it.

A convenient resonator for this purpose is made by drawing the end of a glass tube to such size as will fit tightly into the ear. The other end of the tube is left open. The inclosed air will strengthen very perceptibly, any tone of which the frequency is the same as the frequency of the free vibrations of the air in the tube. This action may be easily perceived if the tube is repeatedly removed from the ear and replaced. For tones of low pitch such a tubular resonator would be unwieldy. In this case it is more convenient to use a hollow *globular* vessel of glass or metal, with a small neck on one side to project into the ear and an opening on the other side several centimeters in diameter. In order to analyze a clang, one after another of a series of such resonators is applied to the ear. Fig. 199 shows an adjustable

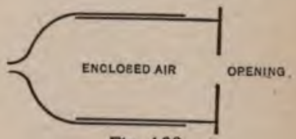

Fig. 199.

resonator designed by Koenig. It consists of two cylindrical brass tubes fitted to one another. The outer tube is contracted to a narrow opening which enters the ear of the observer. The other tube is partially closed by a cap which contains a circular opening 1 to 3 centimeters in diameter.

197. Vowel sounds. — The various vowel sounds are characterized each by one or two tones of *definite pitch*. The characteristic tones of some of the vowels, as determined by Helmholtz, are as follows:

Vowel.	Tone.	Vibration Frequency.*
ṳ as in *rude*	*f*	173
ō as in *no*	*c''*	517
a̤ as in *paw*	*g''*	775
ä as in *part.*	*d*♭'''	1096
ā as in *pay*	*f* and *b*♭'''	346 and 1843
ĕ as in *pet*	*c''''*	2068
ē as in *see*	*f* and *d''''*	173 and 2322

Fig. 200 gives these characteristic tones of the vowels in terms of the ordinary musical notation.

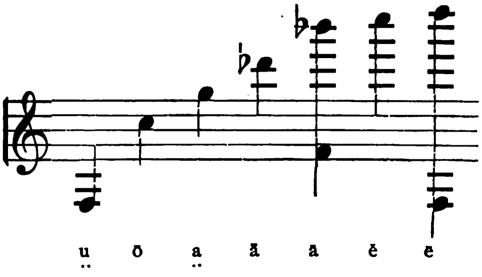

Fig. 200.

In producing a given vowel sound the mouth cavity is shaped so that the contained air has a proper frequency which corresponds to the characteristic tone of the vowel. The sound from the vocal cords is very complex and contains tones of almost every pitch and that particular tone which is in unison with the free or proper vibrations of the air in the mouth cavity is greatly strengthened by resonance. The smooth tone of a singer, however, may not contain the characteristic tone of a vowel, so that this cannot be strengthened by resonance. In this case those overtones of the note which is sung, which are nearest the characteristic tone of a vowel for which the mouth cavity is set, are strengthened, and in this way the vowel sound is (incompletely) characterized. It is a well-known fact that spoken words are much more easily understood than words which are sung. The difference in distinctness is due largely to the imperfect character of vowels when sung. Overtones near a *given pitch* are more widely separated in a note of high pitch than in a note of low pitch, so that the mouth cavity, shaped to give the characteristic tone of a vowel, is less likely to produce the desired effect with high notes than with low. The words of a soprano singer are, in fact, less distinct than the words of a bass singer of similar schooling.

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* Complete vibrations per second.

The proper tones of the mouth cavity for the production of the vowels *ō*, *a̤*, and *ä* may be easily heard by thumping against the cheek when the mouth is prepared to sound those vowels. Helmholtz determined the characteristic tones of the vowels by finding which of a series of tuning-forks placed before the mouth in succession would produce resonance in the mouth cavity shaped to produce a given vowel. He also determined by the help of resonators, as described in the previous article, the actual tones present in the various vowel sounds; and by combining the sounds of suitable tuning forks he was able to imitate successfully these various vowels.

198. The artificial reproduction of speech. *The phonograph.* — Vowel sounds, and indeed all the sounds used in speech, can be reproduced by means of any device which is capable of giving out the necessary combination of simple tones and noises with the proper relative intensities. *The phonograph* is such an instrument.

It consists essentially of a thin diaphragm, to which is fastened a light tool which scratches a minute groove in a rotating smooth cylinder made of a hard, wax-like compound. A sound striking the diaphragm impresses vibrations upon it, and causes the attached tool to cut a groove of varying depth. A *record* of the sound is thus made upon the cylinder. The cylinder is driven with a very nearly uniform motion of rotation by means of clockwork.

To reproduce the sound a round-ended tool, which is attached to the diaphragm, is adjusted to follow this groove and the cylinder is set rotating at its former speed. The varying depth of the groove lifts the tool up and down, causing the diaphragm to vibrate, and the sound is reproduced.

CHAPTER XVI.

THE EAR AND HEARING.

199. The human ear. * — The fact that the various overtones of a clang may be distinctly heard, shows that the conception of *simple* and *compound* vibrations as developed in Arts. 172 and 173 is not merely a mathematical fiction, but that it has real physical significance. This significance is briefly this: namely, that a body, of which the proper vibration frequency is in unison with any simple tone of a clang, is set into violent vibration by the clang. This property has been discussed in the articles on Resonance (Chapter XV.).

It was first pointed out by Helmholtz that our perception of the various simple tones in a clang must depend upon the existence of a series of organs (the end organs of the auditory nerves) in the ear, each of which has a proper vibration frequency and is sensitive (by resonance) to simple tones nearly in unison with it. This action may be illustrated by means of the piano, as follows:

A musical sound of characteristic timbre, for example a vowel sound, is sung loudly against the sounding-board of a piano, of which the damper is raised so as to leave the strings free. Those strings which are in unison with the various simple tones of the clang are set into vibration and we hear a continuation, by the piano, of the vowel sound after the singing ceases. Imagine each string of a piano to be connected to a nerve fiber, and we have an apparatus which would perceive sounds as they are actually perceived by the ear.

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* The anatomy of the ear is too complex to be described here with any fullness. See Helmholtz, "Die Lehre von den Tonempfindungen," pp. 209–250.

The process is as follows: Sound waves enter the ear and strike against the *tympanic membrane*. The vibrations of this membrane, of which the area is about 70 square millimeters, are reduced in amplitude and concentrated* upon another diaphragm, of about 5 square millimeters in area, by the action of a chain of three tiny bones. This second diaphragm covers a small window (the *oval window*) of a bony cavity, called the *labyrinth*, which is filled with a watery fluid. Another opening of the labyrinth, the *round window*, is covered with a diaphragm which is entirely free. The vibrations of the diaphragm in the oval window cause the watery liquid of the labyrinth to surge back and forth between the two windows and through the chambers of the labyrinth. One of these chambers, the *cochlea*, is very long, and is coiled upon itself like a snail shell. The end organs of the auditory nerves are located in membranous structures, which are in part suspended in the watery fluid of the labyrinth, and in part constitute an elastic diaphragm, the *basilar membrane*, which divides the cavity of the cochlea longitudinally. The various shreds of this basilar membrane seem to be the resonating elements of the ear.

200. Persistence of the sensation of sound. — When the stimulation of a nerve ceases, the accompanying sensation continues for a length of time, which depends upon the intensity of the stimulation, and which varies greatly with different nerves. Sensations of light persist much longer than sensations of sound. A periodic light becomes sensibly continuous when the flashes follow one another at the rate of forty per second. A periodic sound — for example, the sound of a tuning fork of high pitch which is shut off from the ear intermittently — has been found by Mayer to become smooth when the fluctuations reach a frequency of one hundred and thirty-five per second.

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* See Helmholtz, *loc. cit.*

201. Beats.*— Consider two simple tones of nearly equal vibration frequencies f and f'. At a certain instant the wave-trains which constitute these tones will be in like phase as they enter the ear. The disturbance produced and the corresponding sensation will then be a maximum.

When the tone of higher pitch has gained half a vibration (or half a wave-length) over the other, the wave-trains will be opposite in phase as they enter the ear, and the sensation will be a minimum. When the higher tone has gained a whole vibration, the waves will again enter the ear in like phase, and the sensation will again be a maximum, and so on. These successive maximum sensations are called *beats*. The number of beats which occur in one second is $f - f'$.

202. Consonance and Dissonance. — An intermittent or fluctuating tone produces an unpleasant sensation which is called *discord* or *dissonance*. A steady tone, or one which fluctuates so rapidly as to give a steady sensation, produces a pleasing effect which is called *concord* or *consonance*. These terms discord or dissonance and concord or consonance are used to express the effects produced by two or more simultaneous tones, that is, they are used to express the *relations of tones*. Nevertheless, the above definitions are physically correct, as will appear in the following discussion.

Consider a tone which is intermittently shut off from the ear, the intermittence beginning at low frequency and increasing to greater and greater frequency. The dissonance at first increases with increasing frequency of intermittence, reaches a maximum, then falls off, and disappears entirely when the frequency of intermittence becomes so great that the sensation of the tone becomes smooth and continuous. The frequency of intermittence for which the dissonance is a maximum, and the frequency of intermittence for which the sensation becomes smooth and continuous depend upon the vibration frequency of the tone which is used, as is shown in the table on page 210.

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* Compare Art. 105.

Vibration Frequency of Intermittent Tone.	Frequency of Intermittence.	
	When Tone Becomes Smooth.	When Discord is a Maximum.
64	16	6.4
128	26	10.4
256	47	18.8
384	60	24.0
512	78	31.2
640	90	36.0
768	109	43.6
1,024	135	54.0

The fluctuations of tone which produce dissonance in music are due to beats. When two tones are in unison they give a smooth sensation. If the vibration frequency of one tone is slowly increased beats occur with greater and greater frequency, the intermittent sound sensation becomes more disagreeable or discordant, and soon reaches a point of maximum discord, after which the discord decreases again. When the beats become sufficiently frequent, the sensation becomes smooth because of the persistence of the sound sensations.

203. Combination tones.—The principle of superposition, namely, that two trains of sound-waves may pass through the same region at the same time without affecting each other (Art. 20), or that an elastic system may perform a series of simple harmonic movements simultaneously without the various vibratory movements being affected by each other (Art. 173) is not strictly true. The failure of this principle of superposition can be shown, mathematically, to be due to what may be termed the incomplete elasticity of ordinary substances; that is, to the fact that the elastic forces brought into play by the sum of two distortions is not exactly equal to the sum of the elastic forces brought into play by the two distortions separately.

Two weak (primary) tones do not sensibly affect each other when they are transmitted simultaneously through the same region of air or when they act simultaneously upon any elastic, or approximately elastic, system such as the transmitting mechanism of the ear. As the primary tones grow louder, however, certain other tones, produced by their mutual action, begin to be heard. These accompanying secondary tones are called *combination tones.* The imperfectly elastic chain of bones in the ear have most to do with the formation of combination tones. Every one perhaps has noticed the harsh rattle in the ear which accompanies very loud tones. This is due also to incomplete elasticity of the chain of tiny bones and is indeed the same thing as the effect known as combination tones, only greatly exaggerated.

Difference tones. — The most prominent combination tone is that of which the frequency is equal to the difference of the frequencies of the two primary tones. This is called the *difference tone* of the *first order.* This difference tone forms difference tones with each of the primary tones, in like manner. These are called difference tones of the *second order;* and so on.

Summation tones. — Less prominent than the difference tones is the combination tone of which the frequency is equal to the sum of the frequencies of the two primary tones. This is called the *summation tone* of the *first order.* Summation tones of the first order form summation tones with each primary tone. These are called summation tones of the *second order.*

Example. — The easiest method, perhaps, to produce an audible combination tone is to use two large steel bars of which the difference in pitch is, say, 100 vibrations per second, and strike them in quick succession with a heavy hammer.

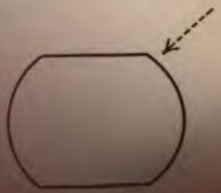

Fig. 201.

A single steel bar of which the section is of the shape shown in Fig. 201, gives a low pitch note when struck on one of the flat faces and a high pitch no[illegible] direction at right angles [illegible] blow in the direction of [illegible] the bar to giv[illegible] both notes simultaneously [illegible] be distinctly heard over a[illegible]

CHAPTER XVII.

MUSICAL INTERVALS AND SCALES.

204. Pitch intervals. — It is shown in the next following article that the consonant relation of two tones depends mainly upon the *ratio* of their vibration frequencies. Two pitch intervals are therefore said to be equal when they are expressed by the same frequency ratio.

Consider a number of tones of which the vibration frequencies are n, an, a^2n, a^3n, etc. The frequency ratio of two successive tones in this series is a. Let their pitch interval be p. The frequency ratio of the first and third tones is a^2, and their pitch interval will be $2p$; the frequency ratio of the first and fourth tones is a^3, and their pitch interval will be $3p$, etc. Let l be the logarithm of a. Then $2l$ is the logarithm of a^2, $3l$ is the logarithm of a^3, etc. Therefore we have for these various pitch intervals the following relation:

Frequency ratios	a	a^2	a^3	a^4	etc.
Logarithms of frequency ratios	l	$2l$	$3l$	$4l$	etc.
Pitch intervals	p	$2p$	$3p$	$4p$	etc.

It will be seen that *the pitch interval between two tones is proportional to the logarithm of their frequency ratio.*

Pitch intervals are ordinarily expressed in terms of frequency ratios. When the *relative magnitude* of a number of pitch intervals is the object of consideration, it is, however, convenient to express the intervals in terms of the logarithms of their frequency ratios. As an example, see the discussion of the tempered musical scale (Art. 215).

205. Consonance and dissonance of compound tones. — The subject of consonance and dissonance of simple tones has been discussed in Art. 202. Tones ordinarily used in music are compound. The fundamental tone usually predominates, in a compound tone, and the overtones 2, 3, 4, 5, 6 and 8 usually occur, decreasing in loudness in the order given. The following discussion of consonance and dissonance *is limited to the influence of these six overtones.*

When two compound tones *A* and *B* are in unison, their respective overtones are in unison also; the combined sound of the two tones is entirely free from roughness due to beats, and the two tones are *completely consonant.* When the two tones are not in unison, then, even if their difference in pitch is so great that the fundamental tones of *A* and *B* do not produce audible beats, some of the overtones of *A* will generally be near enough to some of the overtones of *B* to produce marked roughness in the combined sound of *A* and *B*; that is, to produce distinct dissonance. If the pitch of the tone *B* is slowly raised or lowered, starting from unison with *A*, this dissonance passes through a very marked minimum value every time one (or more) of the overtones of *A* comes into unison with one (or more) of the overtones of *B*. The two tones *A* and *B* are *approximately consonant* when their dissonance thus reaches a minimum value.

The following example will make this clear. Let the tone *B* be a very little higher than *A*. Then the fundamentals and each pair of the overtones produce beats, and the dissonance is great. As the pitch of *B* is raised, this dissonance falls off as each pair of jarring tones becomes more widely separated in pitch. This falling off in the dissonance continues until the fifth overtone of *B* comes to be adjacent to the sixth overtone of *A*. The jarring action of this pair of tones then causes a distinct rise in the dissonance, followed by a rapid fall as the pair come into unison. As the tone *B* continues to rise in pitch, there is a rapid rise in the dissonance, and so on.

The ordinates of the curve in Fig. 202 show the values of the dissonance of two violin tones A and B in so far as the overtones 2, 3, 4, 5, 6 and 8 are concerned. The tone A is kept at a constant pitch and the tone B is first slowly raised in pitch until its vibration frequency is twice as great as the vibration frequency of A and then, beginning with unison with A, slowly lowered in pitch until its vibration frequency is half as great as the vibration frequency of A. The fractions below the line BAB show the values of the frequency ratio B/A for which the various minima of dissonance occur. This curve is adapted from a more complex one by Helmholtz. The numerical evaluation * of dissonance is an approximation.

Remark. — Combination tones have some action in the production of dissonance when two tones are sounded together. The dissonance due to combination tones passes through minimum values for the same pitch intervals as does the dissonance due to overtones.

206. Consonant intervals. — A pitch interval between two tones, for which the tones are approximately consonant, is called a *consonant interval.* Figure 202 shows the various consonant intervals. The following table exhibits the various consonant intervals in the order of the completeness of their consonance, together with their names.

TABLE OF CONSONANT INTERVALS.

1:1	*Unison*	3:5	*Major Sixth*
1:2	*Octave*	4:5	*Major Third*
2:3	*Fifth*	5:6	*Minor Third*
3:4	*Fourth*	5:8	*Minor Sixth*

The consonance of the *octave* is complete. That of the *fifth* is very nearly so.

The bounding of consonant intervals.—Those overtones and combination tones which determine a consonant interval by their coincidence, and which produce the greater part of the dissonance when the interval is slightly out of tune, are said to *bound* the interval.

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* See Helmholtz, "Tonempfindungen," Beilage XV.

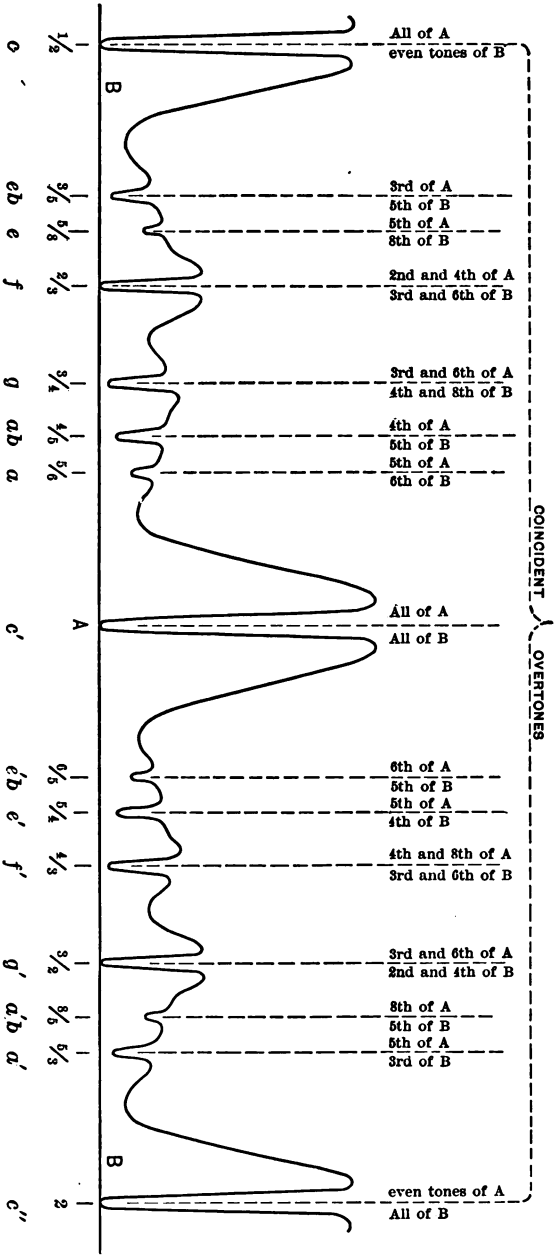

All of A
even tones of B
3rd of A
5th of B
5th of A
8th of B
2nd and 4th of A
3rd and 6th of B
3rd and 6th of A
4th and 8th of B
4th of A
5th of B
5th of A
6th of B
All of A
All of B
6th of A
5th of B
5th of A
4th of B
4th and 8th of A
3rd and 6th of B
3rd and 6th of A
2nd and 4th of B
8th of A
5th of B
5th of A
3rd of B
even tones of A
All of B
COINCIDENT OVERTONES
B
A
B
1/2
3/5
5/8
2/3
3/4
4/5
5/6
6/5
5/4
4/3
3/2
8/5
5/3
2
c
eb
e
f
g
ab
a
c'
e'b
e'
f'
g'
a'b
a'
c''

Fig. 202.

Remark. — The great increase in dissonance, due to a slight error of tuning of a consonant interval, is the basis for the remarkably acute sense which we have of the accuracy of these intervals. This acute sense of pitch of consonant tones has a great deal to do with the effectiveness of consonant intervals in music; for there can be no refinement of musical expression without an acute sense to seize upon it, and it is the ultimate dependence of this acute sense upon the presence of prominent overtones which explains the peculiar musical value of such tones as those of the violin and of the human voice.

207. The variation of the character of the consonant intervals with timbre. — A consonant interval is the more striking in character in proportion as it is more sharply bounded. Therefore the character of a given consonant interval varies with the timbre of the tones used; that is, with the relative loudness of the various overtones.

This is exemplified by clarinet tones, which have only odd harmonics. The consonance of two such tones is most striking when it depends upon coincidences of odd harmonics of both tones, in fact, a consonant interval which depends upon, or would depend upon, the coincidence of even harmonics is bounded only by combination tones. Thus with two clarinet tones the major sixth (3 : 5) is more striking in character than the fourth (3 : 4), and perhaps even as sharply defined as the fifth (2 : 3); while the minor third (5 : 6) and the major third (4 : 5) are ill defined. A major third (4 : 5), formed by a violin tone and a clarinet tone, is very much more striking when the clarinet tone is the lower, so that its fifth overtone coincides with the fourth overtone of the violin, than it is when the violin tone is lower; and a minor third (5 : 6) is much more striking when the violin tone is the lower. A violin tone has both even and odd harmonics.

In the case of pure tones, such as the tones of tuning forks and of wide organ pipes closed at one end, combination tones, only, serve to bound the consonant intervals. With such tones, the octave (1 : 2) is pretty sharply defined, the fifth (2 : 3) less sharply, while the remaining intervals are scarcely bounded at all. Helmholtz, indeed, has found that the sound of two tuning forks is smooth, or consonant, whatever the pitch interval, provided only that the tones are not loud enough to bring out the combination tones strongly.

208. The major and minor accords.—Three tones which form a consonant combination are called an *accord* or *cord.* Thus three tones, of which the vibration frequencies are as 4 : 5 : 6, form an accord.

Any tone of an accord may be replaced by its octave, or may be accompanied by its octave, without greatly altering the character of the accord. This is evident when we consider that no new overtones are introduced into the sound by the octave.

The major accord and its modifications.—The three tones of which the vibration frequencies are as **4 : 5 : 6** constitute what is called the *major accord.* By replacing the first tone (4) by its octave (8), we obtain a modification of this accord, and by replacing the third tone (6) by its lower octave (3), we obtain another modification. The three forms of the major accord are, therefore,

3	4	5		
	4	**5**	**6**	
		5	6	8

The minor accord and its modifications. — The three tones of which the vibration frequencies are as **10 : 12 : 15** constitute what is called the *minor accord.* The interval between the first two tones is a minor third (5 : 6), between the last two tones is a major third (4 : 5), and between the first and last the interval is a fifth (4 : 6); so that the minor accord contains the same consonant intervals as the major accord (4 : 5 : 6). The modifications of the minor accord are

10	**12**	**15**		
	12	15	20	
		15	20	24

The primary forms of the major and minor accords, viz., 4 : 5 : 6 and 10 : 12 : 15, are those in which the three tones are separated by the smallest pitch intervals.

Difference in character of major and minor accords. — The major and minor accords contain the same consonant intervals, and the coincident overtones are identical in the two cases. The combination tones, however, are very different. The following schedule shows the combination tones of the first and second orders.

The major accord.

Primary tones				4	5	6
First difference tones	1	2				
Second difference tones		2	3	4	5	

The minor accord.

Primary tones							10	12		15
First difference tones	2	3	5							
Second difference tones				7	8	9	10	12	13	

This schedule shows that the difference tones of the major accord are exact duplications, or duplications in the lower octaves, of the primary tones. That is, no foreign tones are introduced into the major accord by the combination tones. On the other hand, some of the difference tones of the second order, viz., 7, 8, 9, and 13, which occur in the minor accord, are dissonant, and give to this accord a character very different from that of the major accord.

209. Musical scales. — The successive tones in a melody and the simultaneous tones in harmony are chosen with reference chiefly to their consonance. A collection of the notes which are available for melody or harmony is called a *musical scale*.

The major scale. — Consider a given tone c'. The tones which can be used with c' with musical effect are those designat
$e'\flat$, e', f', g', $a'\flat$, and a' in Fig. 202. Ignoring the
d $a'\flat$, which have low degrees of consonance with

the following series of musical tones, each of which is consonant with c':

Tones	c'	e'	f'	g'	a'	c''	I.
Vibration frequencies	1	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	2	

Remark. — The tone c', with reference to which a series of tones is selected, is called the *tonic* of the series. The tone g', having next to the octave the most complete consonance with c', is called the *dominant*; and the tone f', which is next in order of consonance, is called the *subdominant* of the series.

For purposes of harmony, it is desirable to be able to build major accords (4 : 5 : 6) upon the *tonic*, upon the *dominant*, and upon the *subdominant* of a series of musical tones. Two of these major accords may be built up with the tones in series I., namely, c', e', g' (4 : 5 : 6) and f', a', c'' (4 : 5 : 6). To build a major accord upon g', two additional tones, say b' and d'', are required such that $g' : b' : d'' = 4 : 5 : 6$. Therefore the vibration frequencies of b' and d'' are $\frac{15}{8}$ and $\frac{9}{4}$ respectively. Taking a tone d', an octave below d'', we have the series:

Tones	c'	d'	e'	f'	g'	a'	b'	c''	II.
Vibration frequencies	1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2	

This is the ordinary musical scale, called the *major scale.*

The minor scale. — Choosing, with the help of Fig. 202, the tones *below* c', which are most nearly consonant with c', we have the series:

Tones	c	$e\flat$	f	g	$a\flat$	c'
Vibration frequencies	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	1

or, in order that this series may be more easily compared with I., we may choose all of these tones an octave higher, whence we obtain the following series of musical tones:

Tones	c'	$e'\flat$	f'	g'	$a'\flat$	c''	III.
Vibration frequencies	1	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{8}{5}$	2	

Remark. — This series III. is more melodious when sounded in the order of descending pitch than when sounded in the re- [illegible] of the series are more [illegible] whichever of these tones is sounded first is made correspondingly prominent. The series I. (and also II.) is more melodious when sounded in the order of ascending pitch.

The series III. includes the two minor accords (10, 12, 15) c', $e'\flat$, g', and f', $a'\flat$, c''. To build a minor accord upon g', two additional tones, say $b'\flat$ and d'', are required, such that $g' : b'\flat : d'' = 10 : 12 : 15$, so that the vibration frequency of $b'\flat$ is $\frac{9}{5}$ and the vibration frequency of d'' is $\frac{9}{4}$. Taking a tone d', an octave below d'', we have the series :

Tones	c'	d'	$e'\flat$	f'	g'	$a'\flat$	$b'\flat$	c''	IV.
Vibration frequencies	1	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{9}{5}$	2	

This series of tones is called the *descending* minor scale. For purposes of melody this scale is changed to the following for ascending movements :

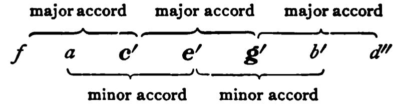
Tones	c'	d'	$e'\flat$	f'	g'	a'	b'	c''	V.
Vibration frequencies	1	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2	

This is called the *ascending* minor scale.*

The scales II. and IV. are better suited to the requirements of harmony than is scale V. The scale II. is suited to harmony in which major accords predominate. It is for this reason called the major scale. The scale IV. is suited to harmony in which minor accords predominate. It is for this reason called the minor scale.

The following schedules exhibit all of the major and minor accords which can be formed of the tones of major and minor scales.

The major scale.



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* Scale IV. is called the descending *melodic* minor scale, scale V. is called the ascending *melodic* minor scale ; a third minor scale which is more largely used than either of these in modern music is the *harmonic* minor scale which is as follows .

c' d' $e'\flat$ f' g' $a'\flat$ b' c''

The use of this scale results in part from employment in music of consonant effects depending upon the seventh harmonic (that is, the harmonic which vibrates seven times as fast as the fundamental).

The minor scale.

major accord major accord
$f \quad a\flat \quad \mathbf{c'} \quad \mathbf{e'\flat} \quad \mathbf{g'} \quad b'\flat \quad d'$
minor accord minor accord minor accord

The tonic accord is shown in each case by the bold-faced type. In the major scale the tonic accord, the dominant accord, and the subdominant accord are major accords. In the minor scale these accords are minor accords.

The naming of consonant intervals. — The intervals between the tonic and the third and sixth tones of the major scale are called the *major third* and *major sixth* respectively. The intervals between the tonic and the third and sixth tones of the minor scale (IV.) are called the *minor third* and *minor sixth* respectively. The intervals between the tonic and the fourth and fifth tones of either scale are called the *fourth* and *fifth* respectively.

Remark. — The tones which are consonant with the tonic c' are called *related tones of the first order.* The tones d' and b' of the major scale and d' and $b'\flat$ of the minor scale which are consonant with g' are called *related tones of the second order* (that is, related to the tonic c').

210. Musical expression. — Expression of any kind depends upon the use of forms which may be readily seized upon and sharply distinguished by the senses. Compare Art. 206, *Remark.* The primary forms of musical expression are *rhythm*, *melody*, *harmony*, and *modulation.* The artistic use of these elements in music is largely traditional and conventional; still the development of musical method has been largely determined by physical facts or laws, and the following articles give brief statements of the physical nature of rhythm, melody, harmony, and modulation.

211. Rhythm. — The rapidity of succession of the tones in music and the manner in which successive tones are set off in groups is called *rhythm.*

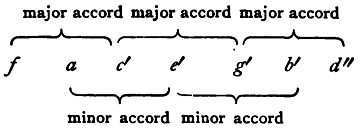
The rhythmic grouping of tones enables a listener to apprehend them more clearly, and therefore permits the effective use of more extended phrases in melody and modulation than would otherwise be possible. Rhythm is also effective in giving expression to vigor or languor according to the rapidity of the movement; and changes of rhythm serve to mark the progress of long compositions, and at the same time to give variety.

212. Melody. — A sequence of tones is called a *melody*. The shades of expression produced by different sequences of tones depend mainly upon the coincidence of overtones of successive notes. When two successive notes have several coincident overtones, the progressive effect of the melody, or, as it is technically called, the *movement*, comes to a momentary stop of a more or less decided character. Thus a repetition of the same tone is a full stop in the movement, and the progressive effect becomes more and more distinct as the consonant relation of successive notes becomes less and less marked. When successive tones are dissonant, the movement is abrupt, and is expressive of sudden emotions, such as surprise. The interval $\frac{15}{16}$ between b' and c'' of the major scale is an exception to this last statement. The note b' seems to serve as a catch note, or a threshold, to be passed in reaching c''.

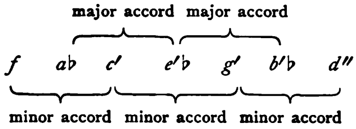
213. Harmony. — The simultaneous use of a number of tones in music is called *harmony*. The shades of musical expression produced by different combinations of tones depend mainly upon the degree of consonance of the combination. Harmony built upon major accords is strikingly different in expression from that built upon minor accords. The first gives an expression of contentment and cheerfulness, while the latter is expressive of discontent and sadness.

214. Modulation. — Two accords are said to be related when they have a common tone or two common tones. A sequence of accords, each of which is related to the one preceding it, is called a *modulation*. The possibilities of modulation in the major and minor scales are exhibited by the following schedule, taken from Art. 209.

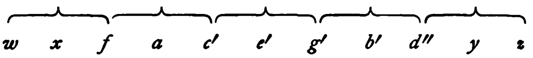
Major scale.


major accord major accord major accord
f a c' e' g' b' d''
minor accord minor accord

Minor scale.


major accord major accord
f a♭ c' e'♭ g' b'♭ d''
minor accord minor accord minor accord

More extended modulations than those exhibited in this schedule require the use of tones related, in the third order, to the tonic c'. Let us consider, for example, an extension in both directions of the modulation of the major scale. We have:



The brackets represent major accords. The tones y and z are not related to c', but they are related to g' exactly as b' and d'' respectively are related to c'. Thus the extension of this modulation upwards leads to a set of tones having a new tonic, namely, g'. In like manner the tones w and x are related to f exactly as f and a respectively are related to c', so that an extension of this modulation downwards leads to a set of tones having a new tonic, namely, f. Such an extended modulation in one direction or the other is called a *change of key*.

In popular music, for example dance music, the modulation is confined to the tonic, dominant, and subdominant accords, which succeed each other periodically. The greater musicians often use very extended and very complex modulations, making use of major and minor accords in one and the same phrase, with occasional dissonances for the sake of contrast.

	Tonic.												Octave.
Number of tone in tempered scale.	1	2	3	4	5	6	7	8	9	10	11	12	13
Log of ratio of tone of tempered scale to the tonic.	0	.0251	.0502	.0753	.1003	.1254	.1505	.1756	.2007	.2258	.2509	.2759	.3010
Nearest tone in major scale.	*c'*		*d'*		*e'*	*f'*		*g'*		*a'*		*b'*	*c''*
Log of ratio of tone of major scale to tonic.	0		.0511		.0979	.1250		.1761		.2219		.2730	.3010
Difference of logs.	0		—.0009		+.0024	+.0004		—.0005		+.0039		+.0029	0

215. The tempered scale. — A great number of distinct tones is required for extended modulation, and it would be impracticable for a player to use a piano or an organ having a separate key (and string or organ pipe) for each tone. This difficulty is overcome, at the expense of accuracy of tuning of the various consonant intervals, by the use of what is called the *tempered scale*. This scale consists of twelve tones (thirteen, counting both end tones) in each octave, the pitch intervals between successive tones being equal. The octave is thus divided into twelve equal pitch intervals. The logarithm of the frequency ratio of each of these intervals is therefore one twelfth of the logarithm of 2, which is the frequency ratio of the octave (compare Art. 204). The table shows the logarithms of the frequency ratios of each tone of the major scale to the tonic, and also the logarithms of the frequency ratios of each tone of the tempered scale to the tonic.

This table shows that the first, third, fifth, sixth, eighth, tenth, twelfth, and thirteenth tones of the tempered scale are very nearly in unison with the successive tones of the major scale. These tones may, in fact, be used for the tones of the major scale; and since the intervals of the tempered scale are all equal, it is clear that *any tone of the tempered scale may be chosen as a tonic, and that the third, fifth, sixth, eighth, tenth, twelfth, and thirteenth tones, counting from the chosen one, constitute a major scale.* All such major scales are equally well in tune. Unlimited modulations may be carried out on this scale inasmuch as any tone reached in a modulation has a group of tones related to it as a tonic. The minor scale may be made up also from the tempered scale and indefinite modulations may be performed.

CHAPTER XVIII.

MISCELLANEOUS PHENOMENA DEPENDING UPON THE REFLECTION, REFRACTION, AND DIFFRACTION OF SOUND. ARCHITECTURAL ACOUSTICS.

216. Echo. — This well-known phenomenon is produced by the reflection of sound. The echo from the side of a large building is very distinct, and the smooth face of a cliff, or a well-defined forest front, may produce an echo sufficiently distinct to repeat words. If the reflecting surface is sufficiently distant, an entire sentence may be repeated by the echo.

An echo grows less distinct the more irregular the reflecting surface, and it becomes an indistinct roar when the reflecting surface is very irregular. With multiple reflections, as in the case of the two walls of a cañon, a sharp loud sound, such as the report of a gun, is prolonged in a manner resembling thunder.

Reflection often produces the effect of an apparent change of direction of a sound, when from any cause the direct waves are masked or diverted, so that the hearer perceives only the reflected wave-trains.

217. The influence of the refraction of sound upon hearing at a distance. — Phenomena due to regular refraction, as sound waves pass from one medium to another, in which the velocity is different, do not occur to ordinary observation. The following phenomena, however, which are due essentially to refraction, are of common occurrence.

The velocity of the wind is usually less near the ground than higher up, and the upper portion of a sound wave *W*, Fig. 203, proceeding against the wind, is retarded. The direction of progression of the wave is thus thrown upwards and the sound tends to leave the region near the ground. When tne wave travels with the wind, the tendency is to concentrate the sound near the ground. It is a familiar fact that it is much more difficult to make one's self heard against than with the wind.

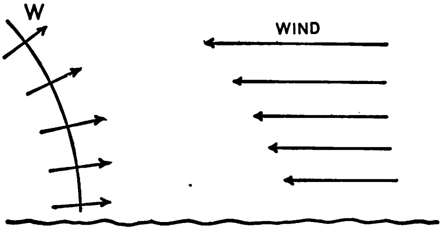


Fig. 203.

Sound travels faster in hot than in cold air. When the air near the ground is warmer than it is higher up, the upper portion of a sound wave is retarded and the sound tends to leave the ground. When the air near the ground is relatively cool, the tendency is for the sound to be concentrated near the ground. The greater distinctness of distant sounds by night than by day is no doubt due largely to this cause.

218. The influence of diffraction upon the sense of direction of a sound. — Our sense of the *direction* of a sound seems to depend, in part, upon diffraction. We have this sense only with complex sounds, not with a sound which consists of a single simple wave-train. The approaching complex waves reach one ear without much obstruction, while the other ear is more or less shaded from them by the head. This shading action is greater the shorter the wave-length, so that different sensations are produced in the two ears. Differences of sensation brought about in this manner are *significant of direction*, and have come to be perceived as such without entering consciousness in any other way.

219. Change of pitch due to relative motion of the sounding body and the hearer. — If a person moves in the direction in which a train of sound waves is traveling, the frequency with which the waves fall upon the ear will be altered and a change of pitch will be produced. Movement of the ear in the direction of progression of a wave-train lowers the pitch, and movement of the ear in the direction opposite to the direction of progression of the wave-train raises the pitch. This effect depends only upon the relative motion of the sounding body and the listener, the above-described effects being produced when the listener is stationary and the sounding body is moving. Thus the whistle of an approaching locomotive is raised in pitch and that of a receding locomotive is lowered in pitch. The effect is especially noticeable to a person on a rapidly moving train who listens to the whistle or bell of a passing locomotive moving in the opposite direction on an adjoining track.

Remark.—The motion of a luminous body towards or away from the observer produces a change in the wave-length of the light given out by the body. Thus when the characteristic grouping of the lines of a star's spectrum leaves no doubt as to the identity of the group of lines to corresponding lines in the solar spectrum, then a slight displacement of the entire group of stellar lines towards the violet or red ends of the spectrum is ascribed to a motion of the star towards us or away from us respectively.

220. Acoustics of the auditorium.* —The simplest auditorium is a level stretch of smooth, hard ground with a single hearer. In this case the sound of the speaker's voice spreads as a hemispherical wave and it decreases in loudness approximately as the inverse square of the distance. If the ground is covered by a large audience the lower edges of the hemispherical sound waves are to a great extent absorbed and thus greatly reduced in intensity; and the sound reaches the distant auditors largely by bending down from above, that is, by diffraction.

The first and most obvious improvement is to raise the speaker above the level of the audience or to arrange the audience upon rising tiers of seats so that each auditor may receive sound by direct radiation, as it were, from the speaker. A second improvement is to place a reflecting wall behind the speaker so as to reflect more of the sound of the speaker's voice towards the audience. A third improvement is to arrange a roof so that the sound which radiates upwards may be reflected towards the audience and utilized. The closed auditorium thus arrived at is approximately perfect in so far as it directs practically the whole of the speaker's voice towards the audience, thus giving the maximum of attainable loudness for a given effort by the speaker. The closed auditorium, however, gives rise to undesirable effects, confusion and distortion, which may be to some extent eliminated by proper design.

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* The material in this and the succeeding articles is adapted from Professor W. C. ne's papers on Architectural Acoustics, *American Architect*, 1900.

Confusion. — The closed auditorium gives rise to an overlapping of the successive sounds in speech or music. This is due to the prolongation of each sound by repeated reflections from the walls and ceiling of the room. If the room is very large these successive reflections of a given sound may be separately audible as a *multiple echo.* If the room is small, or if a large room has its reflecting surfaces broken up by pillars and alcoves, the successive reflections blend into a more or less continuous roar. This effect is called *reverberation.* In some auditoriums this reverberation lasts as long as five or six seconds after a loud sound, and when we consider that there are from fifteen to twenty separate articulate sounds produced by a speaker per second, counting both vowel and consonant sounds, we can understand what a serious matter excessive reverberation may be. In good theaters for the hearing of speech, the reverberation has a duration of from one to two seconds after a loud musical note and in good music halls the reverberation has a duration of from two to three seconds.

Distortion. — Consider the various simple tones which enter into the composition of a compound tone such as a vowel sound. The wave-trains corresponding to these simple tones are reflected from the various walls of a room and there are certain points in the room where one or more of the tones are strengthened while others may be weakened by *interference*. Thus the relative loudness of the various simple tone components of the complex sound is changed and the character of the sound is altered. Furthermore, the air in a room has certain frequencies of vibration of its own, the same as the air in an organ pipe; that is, the air in a room has certain proper tones, and any tone which is in unison with a proper tone of a room is more or less strengthened by *resonance*. Distortion by interference is most prominent in a room with flat reflecting walls, and distortion by resonance is most prominent in small rooms or in a room having alcoves. Distortion is seldom, if ever, a serious matter.

221. The elements of auditorium design. *Loudness.* — In the design of an auditorium provision must be made first of all for directing the sound of the speaker's voice, or the sound of an orchestra, towards the audience. Each auditor should be seated so as to receive sound by direct radiation from the source, and the reflecting surfaces, such as the ceiling and the wall back of the speaker, should be located so that the reflected sound may not reach the auditor more than, say, a twentieth of a second later than the sound which is radiated directly to the auditor; that is, the path of the reflected sound should not be more than about seventeen meters longer than the path of the directly radiated sound, otherwise the successive articulate sounds of a speaker will overlap perceptibly. Furthermore, the reflecting surfaces to be effective must be large in comparison with the wave-lengths of the prominently useful tones in speech and in music.

Multiple echo. — The walls of a large auditorium from which reflected sounds reach the auditors more than a fifteenth or a twentieth of a second after the directly radiated sound, should be broken up by pillars and alcoves so as to avoid sharp echo and the consequent confusing repetition of the articulate elements of speech.

Reverberation. — Let us for brevity call those reflecting surfaces which send sound to the auditors with less delay than a twentieth of a second, primary reflecting surfaces; and those surfaces which reflect sound to the auditors with more delay than a twentieth of a second, secondary reflecting surfaces. All primary reflecting surfaces should be hard and smooth, so as to reflect as much as possible of the incident sound, and all secondary reflecting surfaces should be soft, covered with thick felt for example, so as to reflect as little as possible of the incident sound. The duration of reverberation could of course be reduced to the greatest possible extent by making all reflecting surfaces soft, but a serious sacrifice of loudness would be involved. The control of reverberation is therefore dependent upon the proper treatment of the secondary reflecting surfaces.

222. The equation of decaying sound in a room. — The experimental determination of the absorbing power of various wall surfaces, and the use of such experimental data in the calculation of the duration of reverberation as a guide in the design of an auditorium, depends upon the equation which expresses the intensity i (sound energy per unit volume of the room) of a decaying sound at an instant t seconds after the source of the sound has ceased. This equation is:

$$i = I\epsilon^{-\frac{86a}{V}\cdot t} \tag{31}$$

in which ϵ is the Naperian base and V is the volume of the room in cubic meters. The significance of the quantity a is as follows: The rate at which sound energy is lost in the room is proportional to the intensity of the sound at each instant; the rate at which sound energy would be lost through an open window is also proportional to the intensity of the sound at each instant. Therefore the effect of walls and objects in a room in causing a loss of sound is *equivalent to a certain definite area, a square meters, of open window.* The walls and objects in a room cause a loss of sound [illegible] ore or less distinct ways: (1) As the air particles near a wall oscillate to and fro parallel to the wall sound energy is converted into heat by the friction. (2) When the air particles move to and fro perpendicularly to a porous wall friction occurs in the pores of the wall and energy loss results. (3) A wall partakes more or less of the motion of the contiguous air and if the material of the wall is imperfectly elastic energy losses occur.

Derivation of equation (31).—It is required to find the number of units of sound energy which pass out of a room per second through a window of area a. Let V be the volume of the room and i the sound energy per unit volume at time t seconds after the cessation of the sound source. It is assumed that the sound energy is uniformly distributed throughout the room and that the sound in the neighborhood of a point is traveling, on the average, equally in every direction.

The problem under consideration is simplified if we consider the flow of energy in one direction through a hole in a wall built across the room so as to divide the room into halves. Describe a sphere of radius r with its center at the hole. The area of this sphere is $4\pi r^2$. We may imagine the sound energy per unit volume, i, to be divided into $4\pi r^2$ equal parts, each of which parts is traveling at the normal velocity of sound, 344 meters per second at ordinary room temperature, towards a separate unit of area of the sphere. Draw a diameter to the sphere perpendicularly to the wall. Consider the zone of the spherical surface which lies between θ and $\theta + d\theta$ where θ is the angle shown in Fig. 204. The area of this zone is $2\pi r^2 \sin\theta \cdot d\theta$. Therefore the fractional part, $2\pi r^2 \sin\theta\, d\theta / 4\pi r^2$, of the total sound energy near the hole is streaming towards the various elements of this zone and the component normal to the wall of the velocity of these streams of energy is $344 \cos\theta$. Therefore

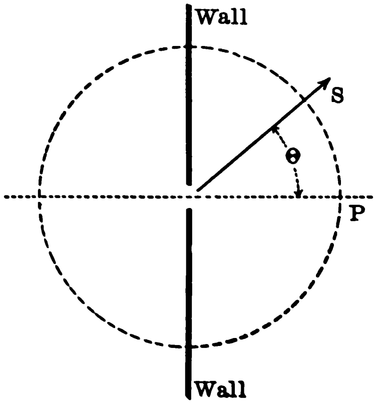

Fig. 204.

$$\frac{2\pi r^2 \sin\theta\, d\theta}{4\pi r^2} \times i \times 344 \cos\theta \times a$$

is the rate at which energy is streaming through the hole towards the various elements of the zone. The total rate at which energy is streaming through the hole from left to right is therefore

$$E = 172ai\int_0^{\pi/2}\sin\theta\cos\theta\cdot d\theta$$

or

$$E = 86ai \tag{32}$$

This rate of flow of sound energy out of a room through a window of area a is equal to the rate, $d(Vi)/dt$, at which the total sound energy in the room, Vi, is decreasing, the room being assumed for the purpose of the immediate discussion to lose sound energy only through the window, therefore

$$\frac{d(Vi)}{dt} = -86ai \tag{33}$$

whence

$$i = I\varepsilon^{-\frac{86a}{V}\cdot t} \qquad \text{(31) bis.}$$

in which ε is the Naperian base, and I is the value of i when t equals zero.

In an actual room the loss of sound energy due to absorption by the walls and objects in the room is sensibly *proportional at each instant to the intensity i of the sound* a that instant. Therefore equation (33) expresses the rate of loss of total sound energy at each instant in an actual room, a being the open window area which is equivalent to the walls and objects in the room in so far as loss of sound energy is concerned, and equation (31) expresses the value of the decaying sound i at time t.

TABLE.

Absorbing power of surfaces.

Character of Surface.	Equivalent Area (P) of Open Window for Each Square Meter of Surface.
Wood sheathing.	0.061
Plaster on wood lath.	0.034
Plaster on wire lath.	0.033
Window glass, single thickness.	0.027
Plaster on tile.	0.025
Brick set in cement.	0.025
Carpet rugs.	0.20 to 0.29
Cretonne cloth.	0.15
Shelia curtains.	0.23
Linoleum, loose on floor.	0.12
Hair felt, 2.5 cm. thick, 8 cm. from wall.	0.78
Audience, compact.	0.96
Audience, compact, per person.	0.44
Isolated man (average).	0.48
Isolated woman (average).	0.54
Upholstered chairs, hair and leather.	0.30
Hair cushions, per seat.	0.21
Elastic-felt cushions, per seat.	0.20
Plain wood settees, per seat.	0.008

223. Absorbing power of various surfaces and objects. — The accompanying table gives the absorbing power of various surfaces and objects as determined by Sabine. These absorbing powers were determined by the methods explained in Art. 226.

To find the open window equivalent of a given surface multiply the area of the surface by p.

To find the open window equivalent of a number of similar objects multiply the number of objects by the open window equivalent of one of the objects.

To find the open window equivalent of all the surfaces and objects in a room add the open window equivalents of each. This gives the value of a in equations (31), (32) and (33).

224. Calculation of duration of reverberation. — The degree of reverberation in a room is expressed by Sabine as the time t_1 required for a musical tone to decay from any initial loudness to a loudness one millionth as great. Let I be the initial loudness, then after t_1 seconds the actual loudness i will have decayed to the value

$$i = \frac{I}{1{,}000{,}000}$$

Substituting this value of i in equation (31) and solving for t_1 we have:

$$t_1 = 0.165 \frac{V}{a} \tag{34}$$

Example. — The New Music Hall in Boston has 1,040 square meters of plaster on lath which is equivalent to 34 square meters of open window, 1,830 square meters of plaster on tile which is equivalent to 45 square meters of open window, 22 square meters of window glass which is equivalent to 0.6 square meter of open window, 625 square meters of wood which is equivalent to 38 square meters of open window, and the full audience including the orchestra is 2,659 persons which is equivalent to 1,169 square meters of open window. Therefore, the value of a for this room with a full audience is 1,286 square meters of open window. The volume of the room is 16,200 cubic meters. From these data equation (34) gives 2.08 seconds as the value of t_1.

Remark 1. — This example serves to show the predominating influence of the audience upon the duration of reverberation. If the seats are hard and smooth the value of a for the empty room would be about a fifth as great as in the above example and the value of t_1 would be about ten seconds. If the seats have cushions the value of a for the empty room would be about half as great as in the above example and the value of t_1 would be about four seconds.

Remark 2. — Consider two rooms A and B built on the same plan and of the same materials, room A being, say, three times as large in every dimension. Then the total absorbing power a of room A will be nine times as great as the absorbing power of room B while the volume of A will be twenty-seven times as great as B. Therefore, the value of t_1 will be three times as great in room A as in room B. That is, reverberation is more and more pronounced the larger the room, and the importance of properly designing large rooms is correspondingly great.

225. Growth of sound in a room. — When a source of sound, an organ pipe, for example, starts in a room, the intensity i of sound in the room increases until the sound is absorbed by walls and objects as fast as it is generated at the source. Let g be the rate at which the sound is generated and $-d(Vi)/dt$ the rate at which the sound is lost by absorption, then when the sound has reached its full degree of loudness we have

$$g = -\frac{d(Vi)}{dt}$$

Substituting this value of $d(Vi)/dt$ in equation (33) and solving for i we have:

$$i_u = \frac{g}{86a} \tag{35}$$

in which i_u is the ultimate loudness or intensity reached by the sound in a room of which the total absorbing power is equivalent to a square meters of open window, and g is the rate at which sound energy is given off by the sounding body.

Remark. — A number n of similar organ pipes similarly blown give off sound energy n times as fast as one pipe, and therefore, the ultimate loudness of the sound produced in a given room is n times as great for the n pipes as it is for a single pipe.

226. Determination of absorbing power. *First method.* — A standard source of sound, for example an organ pipe supplied with air at constant pressure, is set up in a room having hard, smooth walls and provided with a number of adjustible windows. The material of which the absorbing power is to be determined is spread upon the walls or floor and the time that the sound remains audible after cessation of the source is observed. The material is then removed and the windows are opened until the sound remains audible for the same length of time as before. The total area of opening of the windows is then the measure of the absorbing power of the material.*

Second method. Determination of total absorbing power of a room. — The time t'' for the sound of initial intensity I produced by *one* organ pipe to decay to minimum audible intensity i' is observed, and the time t''' for the sound of initial intensity nI produced by n similar organ pipes to decay to minimum audible intensity i' is observed. We then have from equation (31) the two equations:

$$i' = I\epsilon^{-\frac{86at''}{V}} \qquad (a)$$

and

$$i' = nI\epsilon^{-\frac{86at'''}{V}} \qquad (b)$$

Dividing equation (b) by equation (a) we have:

$$1 = n\epsilon^{-\frac{86a}{V}(t'''-t'')}$$

or

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* Some allowance should, of course, be made for the absorbing power of the bare ll or floor which is covered by the material.

$$0 = \log n - \frac{86a}{V}(t''' - t'') \log \epsilon$$

whence

$$a = \frac{V \log n}{86 (t''' - t'') \log \epsilon} \qquad (36)$$

Third method. Determination of absorbing power of the different kinds of wall surfaces and objects. — Suppose for example that there are four kinds of absorbing surface to be determined. Choose four different rooms, A, B, C, and D, in which these four surfaces occur in markedly different proportions.

Let

$$A', A'', A''', \text{ and } A^{\text{iv}}$$
$$B', B'', B''', \text{ and } B^{\text{iv}}$$
$$C', C'', C''', \text{ and } C^{\text{iv}}$$
$$D', D'', D''', \text{ and } D^{\text{iv}}$$

be the respective measured areas of the four classes of reflecting surface in the respective rooms, let p', p'', p''', and p^{iv} be the absorbing power of each kind of surface, and let a', a'', a''' and a^{iv} be the total absorbing powers of the respective rooms. Then

$$a' = A'p' + A''p'' + A'''p''' + A^{\text{iv}}p^{\text{iv}}$$
$$a'' = B'p' + B''p'' + B'''p''' + B^{\text{iv}}p^{\text{iv}}$$
$$a''' = C'p' + C''p'' + C'''p''' + C^{\text{iv}}p^{\text{iv}}$$
$$a^{\text{iv}} = D'p' + D''p'' + D'''p''' + D^{\text{iv}}p^{\text{iv}}$$

The four total absorbing powers a', a'', a''', and a^{iv} may be determined as explained above in the second method. Then these four equations permit the calculation of p', p'', p''', and p^{iv}.

APPENDIX A.

EQUATIONS OF WAVE MOTION.

227. Transverse waves on a stretched string. — Consider a stretched string the position of which when it is quiescent is chosen as the x-axis of reference (Fig. 205). Let Y be the upward displacement of the string at a point distant x from the point which is chosen as the origin of coördinates. Let T be the tension of the string. The element of the string Δx is pulled directly downwards by the portion a of the string by the vertical component $T \cdot dY/dx$ of the tension and the element Δx is pulled upwards by the portion b of the string by the vertical component of the tension at the end b of the element. Now the inclination of the string at end b of Δx is

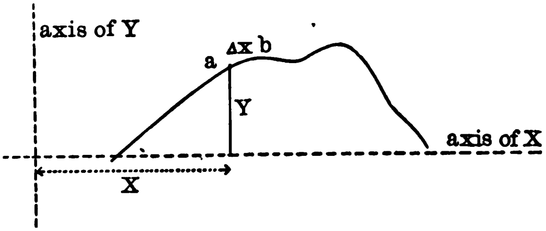

FIG. 205.

$$\frac{dY}{dx} + \frac{d\left(\frac{dY}{dx}\right)}{dx} \cdot \Delta x$$

or

$$\frac{dY}{dx} + \frac{d^2Y}{dx^2} \cdot \Delta x$$

so that the upward pull on Δx is

$$T\left(\frac{dY}{dx}+\frac{d^2Y}{dx^2}\cdot\Delta x\right)$$

and the net upward pull is

$$T\left(\frac{dY}{dx}+\frac{d^2Y}{dx^2}\cdot\Delta x\right)-T\frac{dY}{dx}$$

which is

$$T\frac{d^2Y}{dx}\cdot\Delta x$$

This net upward pull on the element produces an upward acceleration, d^2Y/dt^2, of the element, which acceleration multiplied by the mass of the element gives the value of the net upward pull in dynes. Let m be the mass of the string per unit length, then $m\cdot\Delta x$ is the mass of the element Δx, so that

$$T\frac{d^2Y}{dx^2}\cdot\Delta x=\frac{d^2Y}{dt^2}\cdot m\cdot\Delta x$$

or

$$\frac{d^2Y}{dt^2}=\frac{T}{m}\frac{d^2Y}{dx^2} \tag{37}$$

Write v^2 for T/m or

$$v=\sqrt{\frac{T}{m}} \tag{38}$$

whence equation (37) becomes:

$$\frac{d^2Y}{dt^2}=v^2\frac{d^2Y}{dx^2} \tag{39}$$

The most general possible expression which satisfies (39) is

$$Y=f(x+vt)+F(x-vt) \tag{40}$$

in which f and F signify any functions whatever. The function $F(x-vt)$ represents a wave traveling in the positive direction along the x-axis at velocity v, and the function $f(x+vt)$ represents a wave traveling in the negative direction along the x-axis at velocity v. Let us consider one only of these waves, say the one represented by

$$Y = F(x - vt) \tag{41}$$

In the first place this wave may be of any shape whatever at a given instant, say when $t = 0$, inasmuch as the equation $Y = F(x)$ is the equation to the curve which represents the wave at this instant, and since F signifies any function whatever, therefore $Y = F(x)$ is the equation to any curve whatever.

In the second place this wave retains its shape unchanged with lapse of time, for after the lapse of t' seconds the equation to the curve is $Y = F(x')$ where x' is written for $x - vt'$.

In the third place this wave is traveling to the right at velocity v, for, after t' seconds, we may write x' for $x - vt'$. Then the equation to the wave curve is $F(x')$; that is, the wave curve is of the *same shape as at first when referred to a new origin* which is at a distance vt' to the right of the old origin so that the entire wave curve has moved a distance vt' to the right in time t'.

To show that equation (41) satisfies the fundamental differential equation (39) we need only differentiate (41) twice with respect to x and twice with respect to t thus finding the values of d^2Y/dx^2 and of d^2Y/dt^2 which may then be substituted in (39). Let F'' be the second derivative of F. Then:

$$\frac{d^2Y}{dx^2} = F''$$

and

$$\frac{d^2Y}{dt^2} = v^2F''$$

So that equation (39) is satisfied, and equation (38) shows that the velocity of wave propagation along the string is equal to the uare root of T/m.

228. Transverse waves in an elastic medium.* — Let the rectangle in Fig. 206 represent a portion of a layer of thickness Δx of an elastic medium and let the x-axis of reference be chosen at right angles to this layer. At a given instant imagine this layer to be displaced and distorted by a passing transverse wave so that the rectangle is changed to the dotted rectangle A. Let Y be the upward displacement of the left face, and $Y + \Delta Y$ the upward displacement of the right face of A. Then the distortion of A is a shearing strain, S, equal to $\Delta Y/\Delta x$, and, at the limit, the shearing strain at the left face of A is:

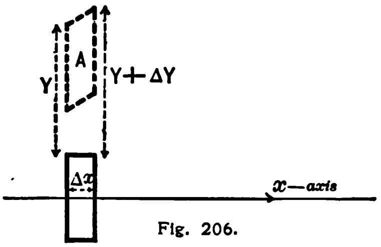

Fig. 206.

$$S = \frac{dY}{dx} \qquad \text{(i)}$$

This shearing strain is accompanied by a shearing stress P, such that

$$P = nS \qquad \text{(ii)}$$

---

* The discussion here given of transverse and of longitudinal waves in an elastic medium assumes a clear understanding on the part of the reader of the elementary theory of elasticity as outlined in Vol. I. of this treatise. It is to be remembered furthermore that there are two principal values of an elastic modulus $\left(\frac{\text{stress}}{\text{strain}}\right)$ according as the material is kept at *constant temperature* or at *constant entropy* during the time that the stress is being applied. By constant entropy is meant that the material is not allowed to give off or to receive heat during the application of the stress. The value of the modulus under constant temperature is called the *isothermic* value of the modulus and the value of the modulus under constant entropy is called the *isentropic* or *adiabatic* value of the modulus. All substances show differences between the isothermic and the isentropic values of their elastic moduli. These differences are most pronounced in gases. In the case of air, for example, the isentropic bulk modulus is 1.41 times as great as the isothermic bulk modulus, as explained in Vol. I. of this treatise. In wave motion the isentropic values of elastic moduli obtain, inasmuch as stresses come and go so rapidly that there is not time enough for the strained portion of the medium to give off or to receive an appreciable amount of heat.

where n is the slide modulus of the substance. Let $S + \Delta S$ be the shearing strain at the right face of A. Then $P + \Delta P = n(S + \Delta S)$ is the shearing stress on that face. Let a be the area of the right and left faces of A. Then $a \cdot \Delta x$ is the volume of A, and $a\rho \cdot \Delta x$ is its mass; ρ being the density of the substance. The force pulling downwards on the left face of A is Pa, and the force pulling upwards on the right face is $(P + \Delta P)a$. The difference $a \cdot \Delta P$ is an unbalanced force which must be equal to the product of the mass of A into its acceleration d^2Y/dt^2. Therefore $a\rho \cdot \Delta x\, d^2Y/dt^2 = a \cdot \Delta P$; or

$$\rho \frac{d^2Y}{dt^2} = \frac{dP}{dx} \tag{iii}$$

Substituting in this equation the value of P from (ii), and then the value of S from (i), we have

$$\frac{d^2Y}{dt^2} = \frac{n}{\rho} \frac{d^2Y}{dx^2} \tag{iv}$$

This equation is identical to equation (37) and therefore all the statements which were made in Art. 227 concerning the transmission of a wave along a stretched string apply also to the transmission of a plane transverse wave in an elastic medium, and the velocity of transmission is $\sqrt{\frac{n}{\rho}}$.

229. Longitudinal waves in an elastic medium. — Let the full line rectangle in Fig. 207 represent a portion of a layer of thickness Δx of an elastic medium, and let the x-axis of reference be chosen at right angles to this layer. At a given instant imagine

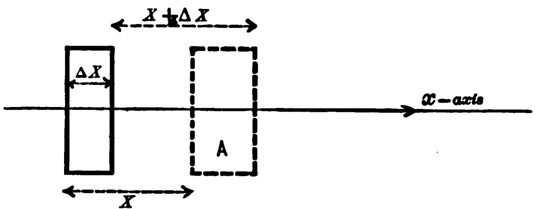

Fig. 207.

this layer to be displaced and distorted by a passing longitudinal wave so that the full line rectangle is changed to the dotted rectangle A. Let X be the displacement of the left face of A, and $X + \Delta X$ the displacement of the right face of A. Then, in a manner precisely similar to the above, it may be shown that

$$\frac{d^2X}{dt^2} = \frac{V}{\rho}\frac{d^2X}{dx^2} \tag{v}$$

in which V is the particular elastic modulus which, multiplied by the strain $\Delta X/\Delta x$, gives the corresponding stress. For liquids and gases V is the bulk modulus.*

Equation (v) is identical to equation (37) and therefore all the statements which were made in Art. 227 concerning the transmission of a wave along a stretched string apply also to the transmission of a plane longitudinal wave in an elastic medium, and the velocity of transmission is $\sqrt{\frac{V}{\rho}}$

Remark 1. — The value of the slide modulus n of a gas or a liquid is zero so that transverse waves are not possible in liquids or gases. For a given solid V and n are as a rule different in value so that longitudinal and transverse waves generally have different velocities in a solid medium.

Remark 2. — In a gas the isentropic bulk modulus V is equal to kp (see Vol. I.) where k is the ratio of the two specific heats of the gas, and p is the pressure. Therefore the velocity of sound in a gas is:

$$V = \sqrt{\frac{kp}{\rho}} \tag{41}$$

for air

$$k = 1.41$$

---

* The modulus V has the following significance for a solid : Consider a strain having but one stretch a. Associated with this strain is a stress of which all three pulls are finite. The pull P, in the direction of a, is taken as the measure of the stress, and we have $P = Va$. It is this stretch a which is represented above by $\Delta X/\Delta x$.

Remark 3. — The ratio p/ρ for a gas is, by Gay Lussac's Law, proportional to the absolute temperature. Therefore the velocity of sound in air is directly proportional to the square root of the absolute temperature. That is:

$$V_t : V_0 :: \sqrt{273+t} : \sqrt{273}$$

or

$$V_t = V_0 \sqrt{\frac{273+t}{273}} \tag{42}$$

in which V_t is the velocity of sound in air at temperature t° C., and $V_0\left(= 331.76 \frac{\text{meters}}{\text{sec.}}\right)$ is the velocity at 0°C.

PROBLEMS.

CHAPTER I.

1. The true period of rotation of Jupiter's second satellite is 85 hours and the velocity of the earth in its orbit is 30 kilometers per second. Find the apparent period of rotation of Jupiter's second satellite when the earth is moving directly away from Jupiter; when the earth is moving directly towards Jupiter.

✓**2.** A hammer strikes a gong at intervals of two seconds. At what intervals, apparently, would the hammer strike to a listener approaching the source of the sound at a velocity of 25 meters per second?

✓**3.** A sounding body makes 200 vibrations per second, that is, the period of the vibrations is $\frac{1}{200}$ second. What would be the apparent period of the vibrations to a listener approaching the sounding body at a velocity of 25 meters per second?

4.* Two bells *A* and *B* 2,000 meters apart are struck at intervals of ten seconds, that is, *A* is struck, then after ten seconds *B* is struck, then after ten seconds *A* is struck again, and so on. An observer hears bell *A*, then after 13.8 seconds he hears *B*, then after 6.2 seconds he hears *A*, and so on. What is the difference of the distances of the two bells from the observer?

---

*It has been proposed to locate *three* bells at known points about the mouth of a harbor and sound them at known intervals of time so that an incoming ship might determine its location by determining the differences of its distances from all three bells. The bells would be submerged and the velocity of sound in water would enter into the calculations.

✓ **5.** Gunners use a stop watch which is started by an observer when he sees the flash of a distant gun and stopped when he hears the report. Find the number of vibrations per second to which the balance wheel of the watch must be adjusted in order that the reading of the "second hand" may indicate hundreds of meters of range directly; the "second hand" being arranged to move one division for each vibration of the balance wheel.

CHAPTER II.

✓ **6.** Find the wave-length of the wave-train produced in air by a body making 16 vibrations per second; by a body making 256 vibrations per second; and by a body making 40,000 vibrations per second.

✓**7.** Twelve drops per second of water fall from a nozzle into a pool of water. The waves produced on the surface of the pool by the drops travel at a velocity of, say, four feet per second. What is the distance between successive waves of the wave-train produced by the drops?

✓**8.** Yellow light has a wave-length approximately equal to 59 millionths of a centimeter. What is the frequency of a wave-train of yellow light?

9. A wave-train is produced on a very long stretched rubber tube by moving the end of the tube up and down at a definite frequency. How many times as much energy will be transmitted along the tube per second if the end of the tube is moved up and down through twice as great a distance as before and at three times the frequency?

10. Given two oppositely moving similar trains of waves A and B, Fig. 208; $a = 1\frac{1}{2}''$, $b = 2\frac{1}{2}''$ and $c = 5\frac{1}{2}''$. Draw these two wave-trains with the point p directly over the point p' and draw the resultant of the two trains for this position. Make five additional drawings showing A and B and their resultant $\frac{1}{8}$ of a iod later, $\frac{2}{8}$ of a period later, $\frac{3}{8}$ of a period later, $\frac{4}{8}$ of later, and $\frac{5}{8}$ of a period later, a period being the time required for the waves to travel over the distance $b + c$.

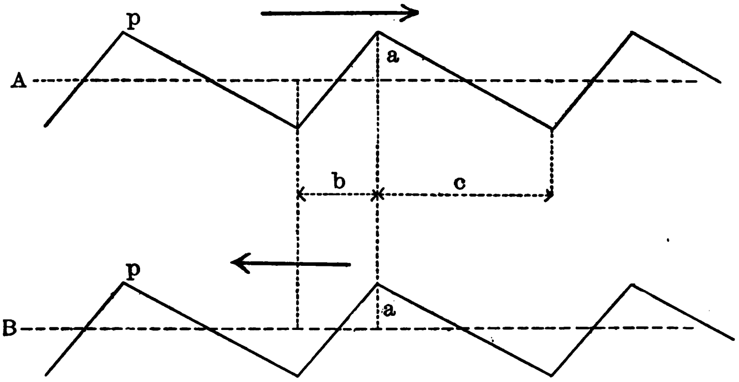

Fig. 208.

11. A train of sound waves coming from a distant body vibrating 256 times per second, strikes a wall perpendicularly and is reflected. A stationary wave-train results. At certain distances from the wall, or at certain points, *A*, the air is subjected alternately to compression and rarefaction but does not move; and at certain distances from the wall, or at certain points *B*, the air moves to and fro but is not condensed or rarefied. What are the points *A* called, and what are the distances of this series of points from the wall? What are the points *B* called, and what are the distances of this series of points from the wall? To which series of points does the surface of the wall belong?

12. The ear is affected only by variations of pressure, not by air movements. At what distance, or distances, from a reflecting wall will the sound of a distant body become inaudible when the body makes 125 vibrations per second and the sound strikes the wall perpendicularly?

CHAPTER III.

13. A plane mirror rotates about an axis that lies in the plane of the mirror. Show that a beam of light reflected from the mirror turns about the axis of the mirror at an angular velocity twice as great as the angular velocity of the mirror.

14. Draw an arc of a circle of 10 cm. radius, length of chord being 14 cm. Consider this circular arc as a concave mirror, like *MM*, Fig. 29, and draw a straight line parallel to *WW* at a distance of 2 cm. behind the central portion of *MM*. Let this line represent the position which a plane wave coming from the right would have reached at a given instant if it had not encountered the mirror. Construct the reflected wave-front at the given instant.

15. Draw the arc of a circle 8 cm. radius to represent a concave mirror of small aperture and find by construction the conjugate to a point which is 12 cm. from the mirror and 2 cm. above the axis. Find by construction the conjugate to a point which is 6 cm. from the mirror and 2 cm. above the axis. Find by construction the conjugate to a point which is 2 cm. from the mirror and 2 cm. above the axis.

16. A concave mirror of which the radius of curvature is 80 cm. forms an image of an object. (*a*) The object is 20 cm. in diameter and 200 cm. distant from the mirror. Find distance of image from mirror and find diameter of image. (*b*) The object is 5 cm. in diameter and 50 cm. from the mirror. Find distance of image from mirror and find diameter of image. (*c*) The object is 5 cm. in diameter and 30 cm. from the mirror. Find distance of image from mirror and find diameter of image. Specify in each case whether the image is real or virtual, erect or inverted.

Remark. — In equation (5) Art. 39, a and b are both positive when they both refer to real foci. When a or b is negative the corresponding focus is virtual and is behind the mirror. That is, a and b when positive are measured towards the front of a concave mirror, when one of them is negative it is measured towards the back of the concave mirror.

The line *WW*, Fig. 209, represents the position which would ched by the wave *CD at a given instant* were it not for the transition from glass to air. Make a drawing showing Huygens' construction for the refracted wave in air at the given instant the index of refraction of the glass being 1.50, and the angle θ being 30°.

18. Make a drawing showing Huygens' construction for the refracted wave in air at the given instant when the angle θ, Fig. 209, is equal to 60°, other data being as in problem 17.

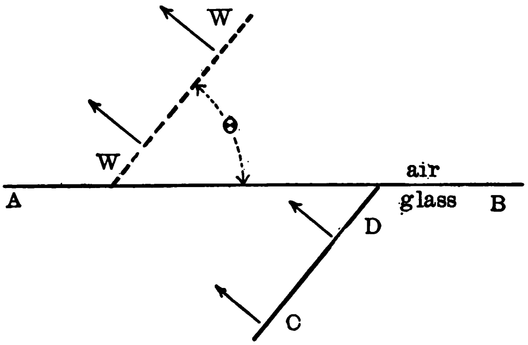

Fig. 209.

19. Prove that the curve *CC*, Fig. 44, is a circle, μ being a constant.

Chapter IV.

20. A lens is made of glass of which the refractive index is 1.6. The radius of curvature of one surface of the lens is 20 centimeters, and the other surface of the lens is flat. What is the principal focal length of the lens?

21. A converging lens has a focal length of 20 centimeters. An object is placed at a distance of 30 centimeters from the lens. Find the distance of the image from the lens and the ratio of diameter of object to diameter of image. Is the image real or virtual, erect or inverted?

22. An object is placed 15 centimeters from a converging lens of which the focal length is 20 centimeters. Find the distance of the image from the lens, and the ratio of diameter of object to diameter of image. Is the image real or virtual, erect or inverted?

23. An object is placed 15 centimeters from a diverging lens, of which the focal length is 20 centimeters. Find the distance of the image from the lens and the ratio of diameter of object to diameter of image. Is the image real or virtual, erect or inverted?

24. The diameter of the object in problems 21, 22 and 23 is 8 cm. Draw to scale the geometrical construction locating the points of the image corresponding to extreme top and bottom of object in each case.

25. A photographic lens of 15 cm. focal length gives sharp images of very distant objects upon the ground glass of the camera when the slider carrying the lens is at marked position. Calculate the distance from this mark to the slider in order that an object two meters from the lens may be sharply focussed on the ground glass.

Chapter V.

26. A projection lantern takes a transparent slide 7 × 7 cm. The focal length of the lantern objective is 20 cm. It is desired to project on the screen an image of the slide two meters square. Required the distance from the lens to the screen.

27. A projection lantern is to be used at a distance of 3 m. from a screen and it is desired to enlarge a 7 × 7 cm. slide to the size of 1 meter square. What focal length objective is required?

28. A pocket magnifier has three separate lenses, of which the focal lengths are 1, 2, and 4 cm. respectively. What is the magnifying power of each when the eye is accommodated for a distance of 25 cm.? When the eye is accommodated for parallel rays?

29. A compound microscope has an objective of which the focal length is 2 mm., an eye-piece of which the focal length is 10 mm., and the distance from center of objective to the plane of the image (*b*, Fig. 98) is 150 mm. Calculate the magnifying power of the instrument. What would the distance *b*, Fig. 98, have to be to give the same magnifying power with an objective 25 mm. focal length, using the same eye-yiece?

30. The object glass of the great telescope of the Lick Observatory has a focal length of 1,500 cm. What is the magnifying power of this telescope when an eye-piece of 2 cm. focal length is used?

CHAPTER VI.

31. A lens is 2 cm. free diameter and 12 cm. focal length. What is its numerical aperture? The object glass of the great telescope of the Lick Observatory is 92 cm. free diameter and 1,500 cm. focal length. What is its numerical aperture?

32. A photographic objective of which the focal length is 15 cm. forms a satisfactory image over the whole of a photographic plate 16 x 22 cm. What is the field angle of the lens?

33. The angular diameter of the sun as seen from the earth is a half degree. What is the diameter, in centimeters, of the image of the sun formed by the object glass of the great telescope of the Lick Observatory?

34. A photographic lens with a numerical aperture of $\frac{1}{6}$ gives a good photograph with, say, $\frac{1}{50}$ second exposure. What exposure would be required with a lens working with a numerical aperture of $\frac{1}{16}$? With a lens working with a numerical aperture of $\frac{1}{32}$?

CHAPTER VIII.

35. A whistle gives a sound of which the wave-length is 5 cm. Sound from the whistle issues through two holes in a wall *OO'*, Fig. 110. The holes are 60 centimeters apart. A person starting from a point *C*, Fig. 110, ten meters from the wall walks along the line *Cp*. How far from *C* will the man be when the sound of the whistle becomes inaudible?

36. A glass plate 0.0004 cm. thick is illuminated by a beam of light making an angle of 30° to the normal to the plate. The index of refraction of the glass is 1.5. What wave-length of the light will be strengthened by interference as explained in Art. 106?

37. The distance *b*, Fig. 116, is 200 cm. and the wave-length of the wave-train *TT* is 50×10^{-6} cm. What is the diameter of the tenth half period zone in the plane *AB*?

38. The action of a diffraction grating is sometimes described in the following terms, referring to Fig. 126: The wavelets passing through the slits *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, and *i* may be divided up into sets, each set having an enveloping plane, and the wavelets of this set coalesce so as to form a plane wave the position of which is given by this enveloping plane.

This method of analyzing the action of a grating is very instructive and to understand the method one must make a drawing.

A train of sound waves *TT*, Fig. 126, of 10 centimeter wave-length, strikes a grating *AB* formed by ten boards each 38 centimeters wide with slits between, each slit being 2 centimeters wide. Let the sets of wavelets which at a given instant passed out from the various slits be called wavelets No. 1, the set of wavelets next following No. 2, the set next following No. 3, and so on. Consider the instant that wavelets No. 8 are at a distance of 100 cm. from their respective slits. Make a drawing to scale showing the grating and the following wavelets: Wavelet No. 8 from the middle slit *e*, wavelets No. 7 and No. 9 from slits *d* and *f*, wavelets No. 6 and No. 10 from slits *c* and *g*, wavelets No. 5 and No. 11 from slits *b* and *h*, and wavelets No. 4 and No. 12 from slits *a* and *i*; and draw the planes which envelop the two sets of wavelets thus drawn.

39. The distance from center to center of the slits in the grating *AB*, Fig. 126, is 0.0003 cm., the light *TT* is white light, and the focal length of the lens *LL'* is 50 cm. Calculate the distances from the point *F* to *F'* and *F''* and to *F'''* and *F''''* for red light ($\lambda = 40 \times 10^{-6}$ cm.) and for violet light ($\lambda = 75 \times 10^{-6}$ cm.).

Chapter IX.

40. A lamp gives sufficient illumination for easy reading at a distance of 60 cm. How many similar lamps must be placed side by side to permit easy reading at a distance of 240 cm., ignoring the light reflected from the walls of the room?

41. A lamp is placed in a room of which the walls are perfectly black, reflecting no light, and a small piece of paper has a certain mean degree of illumination at all the points in the room. How many times will the mean illumination be increased if the walls are covered with a paint which reflects half the light which falls on it? If the walls are covered with a paint that reflects nine tenths of the light which falls on it?

Ans. (*a*) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \cdots$
(*b*) $1 + \frac{9}{10} + \frac{81}{100} + \frac{729}{1000} + \frac{6561}{10000} + \cdots$.

42. The arc lamp used in a search light has 10,000 true candle power and two thirds of this light is reflected by a parabolic reflector giving a beam having a divergence* of 2° or filling a cone of which the angle is 2°. What is the candle power of this beam?

43. Let the brightness of daylight with the sun in the zenith be taken as unity. What is the brightness of daylight when the altitude of the sun is 75°, 60°, 45°, 30°, and 15° above the horizon respectively, ignoring increase of atmospheric absorption with increase of zenith distance of the sun.

Remark. — The relative brightness of daylight is inversely proportional to the area of country covered by a beam of sunlight of, say, one square kilometer in sectional area, and this is inversely proportional to the cosine of the sun's altitude above the horizon.

---

* The divergence of the reflected beam, assuming the reflector to be of perfect shape, is equal to the angle subtended by the luminous area of the lamp as seen from the vertex of the mirror.

44. A certain photographic lens gives a good photograph with an exposure of $\frac{1}{60}$ second when the sun is 75° above the horizon. What exposure would be required with the same lens when the sun is 5° above the horizon, ignoring increase of atmospheric absorption with increase of zenith distance of sun.

CHAPTER XI.

45. A plate of Iceland spar, 2 mm. thick, with the optic axis parallel to the surfaces of the plate, is placed in a polariscope with crossed Nicol prisms. The beam of polarized light consists of a bundle of parallel rays. What wave-lengths between 39×10^{-6} and 75×10^{-6} cm. are cut off by the analyzer and what wave-lengths pass through the analyzer with full intensity? Assume ordinary and extraordinary indices of refraction to be the same for all wave-lengths, namely 1.658 and 1.486, respectively.

46. The intensity of the light which is transmitted through a polariscope with parallel Nicols is taken as unity. What is the intensity of the light when the analyzer is turned 15°, 30°, 45°, 60°, 75° and 90° respectively from the parallel position?

47. A solution of cane sugar in a tube 50 cm. long rotates the plane of polarization of sodium light through 8°.31 of angle. What is the amount of sugar in grams in each cubic centimeter of the solution?

CHAPTER XII.

48. Plot curves on cross-section paper, using λ as abscissæ and E as ordinates, showing relative intensities of the various wave-lengths in normal radiation at absolute temperatures of 1,200° C., 1,800° C. and 2,400° C. Find for each curve the ratio, area under curve between $\lambda = 39 \times 10^{-6}$ cm. to $\lambda = 75 \times 10^{-6}$ cm. divided by total area under curve.

NOTE. — The ratio here required may be most easily calculated with the aid of equation (26). The ratio in question has a value of 0.00603 at 1,800° C. absolute.

49. Find the area of plaster-on-tile wall which must be covered by hair-felt 2.5 cm. thick hung 8 cm. from the wall in order to reduce t_1 to 2 seconds in the New Boston Music Hall where there is half an audience present; the total absorbing power of each unoccupied seat being taken as equivalent to 0.21 square meters of open window.

50. The Packer Memorial Chapel of Lehigh University has a volume of 10,650 cubic meters. In this chapel there are 157 square meters of window glass, 1,200 square meters of wood sheathing, 1,940 square meters of hard plaster (including the hard tile floor), 375 square meters of extra heavy curtain ($p = 0.40$), 80 seats of cushions, 45 square meters of openings into organ, ante-rooms, etc., and plain wood settees for 770 persons. Calculate t_1 with and without audience.

Ans. 2.57 seconds and 5.12 seconds.

Calculate the area of wood sheathing which, in the Packer Memorial Chapel, must be covered with hair-felt 2.5 cm. thick placed 8 cm. from the surface of the wood in order to reduce t_1 to 1.5 seconds with full audience.

Ans. 679 square meters.

51. Assuming that the absorbing power of hair-felt 2.5 cm. thick laid close against a wall is 0.50, calculate the amount of wood surface (and of hard wall surface in addition if necessary) which must be covered in the Packer Memorial Chapel to reduce t_1 to 1 ½ seconds with full audience.

INDEX.

NUMBERS RELATE TO PAGES.

ABBE'S sine law, 74.
Aberration, chromatic, 76.
    spherical, 73.
Absorbing power, determination of, for sound, 236.
    power of wall surfaces, for sound, 234.
    table, 233.
Absorption and emission of radiation, 168.
    coefficient of, 174.
    color, 172.
Accommodation of the eye, 61.
Accords, the major and minor, 217
Achromatic lens, the, 76, 90.
Air columns, vibration of, 191.
Amici's principle, 45.
Amplitude, definition of, 179.
    of wave train, 13.
Anastigmatic lens system, 77.
Anomalous dispersion, 175.
Aplanatism, 73.
Apparent size of objects, 62.
Apochromatic microscope objective, 87.
Astigmatic pencil, 24.
Astigmatism of a lens, 77
    of the eye, 62.
Auditory nerves, 2.
Ångstrom unit, the, 120.
Audibility limits of pitch, 181.
Auditorium, acoustics of, 228.
    decay of sound in, 231.
    growth of sound in, 235.
    reverberation in, 234.
Auditorium design, 230.

BEATS, 102–209.
Black bodies, 170.
Bolometer, the, 172.
Bouguer's principle, 124.
Brightness, 133.
    of homogeneous light, 133.
    intrinsic, 127.
    sensations of, 133.
Bunsen's photometer, 125.

CAMERA, photographic, 63.
Candle power, mean spherical, 128.
    the standard, 121.
Carcel lamp, the, 122.
Chladin's figures, 187.
Chromatic aberration of lens, 76.
Clang, definition of, 180.
Clarinet, the, 196.
Collimating lens of spectroscope, 92.
Color, 133.
    by absorption, 172.
    blindness, 136, 139.
    blindness, Holmgren's test for, 141.
    cause of in natural objects, 135.
    contrasts, 138.
    mixing, 136.
    sensations of, 133.
    surface, 171.
    top, the, 136.
    Young-Helmholtz theory of, 137.
Colors, complementary, 137.
    of homogeneous light, 134.

- Color, mixed light, 135.
  - of thin plates, 102.
- Combination tones, 210.
- Compensated lens systems, examples of, 84.
- Complementary colors, 137.
  - radiations, 166.
- Composition of light, 132.
  - of radiation, determination of, 172.
- Compound modes of vibration, 186.
- Concave grating, the, 120.
- Concord of compound tones, 213.
  - of simple tones, 209.
- Conjugate foci, see lens.
- Consonant intervals, naming of, 221.
  - intervals, bounding of, 214.
  - intervals, table of, 214.
  - intervals and timbre, 216.
- Consonance of compound tones, 213.
  - of simple tones, 209.
- Contrasts, color, 138.
- Converging lens, the, 48.
- Cornet, the, 196.
- Corpuscular theory of light, 3.

- DAMPING of vibrations, 201.
- Decaying sound in room, 231.
- Diaphragms, vibrating, 199.
- Dichroic vision, 136, 139.
- Difference tones, 211.
- Diffraction, definition of, 106.
  - examples of, 110, 112, 113, 114.
  - grating, the, see grating.
  - spectrum, 117.
- Discord of compound tones, 213.
- Discord of simple tones, 209.
- Dispersion, anomalous, 175.
  - definition of, 89.
  - Helmholtz's theory of, 174.
- Dissonance of compound tones, 213.
  - of simple tones, 209.
- Distortion of image by a lens, 79.
- Diverging lens, the, 48.
  - ble refraction, 148.
  - refraction theory of, 149.
  - nant of musical scale, 219.

- EAR, the human, 207.
- Echo, 226.
- Elastic-solid theory of light, 3.
- Electro-magnetic theory of light, 4.
- Emission and absorption of radiation, 168.
  - selective, 168.
- End organs of nerves, 1.
- Ether, the luminiferous, 3.
- Expression, musical, 221.
- Eye, the, 61.
  - accommodation of, 61.
  - astigmatism of, 62, 78.
  - imperfections of, 62.
- Eye-pieces, examples of, 85.
- Eye-piece, Huygens', 85.
  - Ramsden's, 85.

- FIELD angle of lens, 73.
- Field curvature of a lens, 81.
- Fizeau's method for light velocity, 7.
- Flame, the sensitive, 102.
- Flicker photometer, the, 128.
- Fluorescence, 175.
- Focal length, see lens.
- Foci, real and virtual, 50.
- Focus, see lens.
- Forced vibrations, 201.
- Foucault's method for light velocity, 9.
- Fourrier's Theorem, 179.
- Fraünhofer's lines, 95.
- Free vibrations, 201.
- Fresnel's mirrors, 101.
- Frequency, definition of, 178.

- GLOW-LAMP standard, the, 123.
- Grating, the concave, 120.
  - the diffraction, 114.
  - the reflection, 118.
  - spectrometer, the, 119.
  - the transmission, 118.
- Green blindness, 136, 139.

- HALF-PERIOD zones, 106, 109.
- Harmony, 222.
- Hearing at a distance, 226.
- Hefner-Altenek lamp, the, 123.
  - n test fo ss, the,

Homocentric pencil, 23.
Homogeneous immersion microscope objective, 88.
  light, brightness of, 133.
  light, color of, 134.
  light, definition of, 89, 90.
Huygens' construction for wave front, 22.
  eye piece, 85.
  principle, 21.
  principle application to reflection and refraction, 27.
  theory of double refraction, 149

ILLUMINATION, intensity of, 127.
Image, magnification of, 55.
Images, formation of by lens, 54.
Impressed vibrations, 201.
Index of refraction, definition of, 26.
Interference, by thin plates, 102.
  example of, 98.
  fringes, 98, 99.
  fringes, colored, 100.
  fringes in sound, 101.
  fringes, production of, 100.
  from similar sources, 97.
Interferometer, the, 104.
Intervals, consonant, naming of, 221.

KEY, musical, change of, 223.
Kundt's experiment, 197.

LANTERN, the projection, 63.
Lens, the, 48.
  achromatic, the, 76, 90.
  astigmatism of, 77.
  chromatic abberation of, 76.
  conjugate foci of, 50.
  converging, the, 48.
  curvature of field of, 81.
  distortion of image by, 79.
  diverging, the, 48.
  field angle of, 73.
  formation of images by, 54.
  imperfections, 70.
  numerical aperture of, 70.
  orthoscopic, 79.
  principal focal length of, 49.
Lens, principal focal planes of, 49.
  principal focus of, 49.
  rectilinear, 79.
  resolving power of, 71.
  spherical aberration of, 73.
  systems, 56.
  systems, compensated, examples of, 84.
Lenses, photographic, 86.
  wide-angle, 82.
  wide-aperture, 82.
Light, composition of, 132.
  defined as a proper stimulus, 1.
  defined as a sensation, 1.
  distribution of about a lamp, 127.
  the corpuscular theory of, 3.
  ordinary, 143, 145.
  polarized, 143, 144.
  plane polarized, 145.
  circularly polarized, 145.
  elliptically polarized, 145.
  elastic-solid theory of, 3.
  electromagnetic theory of, 4.
  wave theory of, 3.
  homogeneous, 89.
  nonhomogeneous, 89.
  standards, 121.
  transmission of, 2.
  velocity of, 7.
Lissajou's figures, 184.
Lloyd's mirror, 101.
Loudness of tones, 181.
Luminescence, 176.
Lummer-Brodhun photometer, the, 126.

MAGIC LANTERN, the, 63.
Magnifying glasses, examples of, 85.
  glass, the, 63.
Major accord, 217.
  scale, the, 218.
Mean-spherical candle power, 128.
Melody, 222.
Methven screen, the, 122.
Michelson and Morley's interferometer, 104.
Micron, the, 120.
Microscope objectives, compensated, 88.

Microscope, resolving power of, 72.
  the compound, 65.
  the simple, 63.
Minor accord, 217.
Mirror, the spherical, 34.
Modes of vibration of elastic systems, 186.
Modulation, 222.
Morley and Michelson's interferometer, 104.
Musical scales, 218.

NERVES, auditory, 2.
  optic, 1.
  sensory, 1.
Newton's rings, 104.
Nicol prism, the, 154.
Nodal points of an elastic system, 60.
Node, definition of, 16.
Nodes in vibration systems, 186.
Noises, physical nature of, 180.
Non-homogeneous light, definition of, 89, 90.
Numerical aperture of lens, 70.

OBJECTIVES, compensated, for microscopes, 87.
  compensated for telescopes, 84.
Opera glass, the, 67.
Optic axis, definition of, 149.
Optic nerves, 1.
Organ pipes, 195.
Orthoscopic lens, theory of, 81.
  system, 80.

Particle, vibration of, 177.
Pencil, astigmatic, 24.
  homocentric, 23.
  of rays, definition of, 23.
Period, definition of, 178.
Phase, change of at reflection, 20.
  of wave, 13.
Phonograph, the, 205.
Phosphorescence, 175.
Photographic camera, the, 63.
  lenses, 86.
Photometer, Rumford's, 125.
  the Bunsen, 125.
  the flicker, 128.
Photometer, Lummer-Brodhun, 126.
  the shadow, 125.
Photometry, simple, 121.
  spectro-, 121.
Pitch, change of, due to motion, 227.
  determination of, 182.
  intervals, 212.
  limits of audibility, 181.
  of tones, definition of, 181.
  standards of, 182.
Polariscope, the, 156.
Polarization and double refraction, 143.
  by reflection, 146.
  of skylight, the, 163.
  rotation of plane of, 162.
Polarized light, 143, 144.
Prevost's principle of exchanges, 165.
Principal planes of a lens system, 57.
  planes, inverse, of a lens system, 59.
Prism, minimum deflection of, 40.
  the, 40.
  the Nicol, 154.
Problems, 245.
Proper vibrations, 201.

RADIANT heat, chemical effects of, 164.
  heat, definition of, 164.
  heat, luminous effects of, 164.
Radiation, 164.
  composition of, 166.
  normal, 165.
  normal, formula for, 166.
Radiations, complementary, 166.
Radiometer, the, 173.
Radiation, intensity of, 164.
Ramsden's eye-piece, 85.
Ray, the, of light, 22.
  extra-ordinary, 149.
  ordinary, 149.
Rays, infra-red, 89.
  ultra-violet, 89.
Rectilinear lens system, 80.
Red blindness, 136, 139.
Reed pipes, 195.
Reflection, regular and diffuse, 26.
  application of Huygens' principle to, 27.

Reflection, at a plane surface, 28, 29, 30.
  at a curved surface, 31.
  selective, 168.
  total, 41.
  with change of phase, 20.
  without change of phase, 20.
Refraction, definition of, 26.
  of sound, 226.
  application of Huygens' principle to, 27.
  at a plane surface, 39, 40, 41, 42.
  at a spherical surface, 44.
  of narrow pencil at spherical surface, 46
  double, 148.
  double, theory of, 149.
Refractive index, determination of, 96.
  index, definition of, 26.
Resolving power of lens, 71.
  power of microscope, 72.
Resonance, 202.
Resonator, 202.
Resonators, use of for analysis of clangs, 203.
Reverberation in an auditorium, 234
Rhythm, musical, 221.
Rumford's photometer, 125.

SACCHARIMETER, the, 162.
Scales, musical, 218.
Scale, the tempered musical, 224.
Selective action in radiation, 168, 169.
Sensation, 1.
Sensations of brightness, 133.
  of color, 133.
Sense of direction of sound, 227.
Sensitive flame, the, 102.
Sensory nerves, 1.
Shadow photometer, the, 125.
Similar sources, definition of, 97.
Simple modes of vibration, 186.
Size, apparent, of objects, 62.
Skylight, polarization of, 163.
Soap-film, colors of, 102.
Sound, decay of in a room, 231.
  defined as a proper stimulus, 2.
  defined as a sensation, 2.
Sound, sensation, persistence of, 208.
  transmission of, 2.
  wave theory of, 3.
  velocity of, 5.
Sodium lines, reversal of, 95.
Spectrometer, the, 95.
  the grating, 119.
Spectrophotometric measurements, examples of, 130.
Spectrophotometer, the, 96, 129.
Spectrophotometry, 121.
Spectroscope, the, 92.
  direct vision, 91.
Spectrum, definition of, 89.
  the diffraction, 117.
  bright line, 93.
  continuous, 92.
  dark line, 94.
  infra-red, 89.
  the solar, 95.
  ultra-violet, 89.
  visible, 89.
Spherical aberration of lens, 73.
Sky glass, the, 67.
Stationary wave-train, 14.
Stigmatic lens system, 77.
Stimulus, nerve, 1.
  proper, 1.
String, transverse vibration of, 188.
Subdominant of musical scale, 219.
Summation tones, 211.
Superposition, the principle of, 14.
  of vibrations, 179.
Surface color, 171.

TABLE of wave-lengths, 119.
Telescope, the, 61.
  the erecting, 67.
  use of for sighting, 68.
Timbre, definition of, 183.
Tone, compound, 180.
  musical, 180.
  simple, 180.
Tonic of musical scale, 219.
Total reflection, 41.
Tourmaline crystals, optical behavior of, 144.

Trains of waves, 11.
Transmission, selective, 168.
Trichroic vision, 136.
Tuning fork, the, 199.

UNDULATORY theory, see wave theory.

VELOCITY of light, 7.
  of sound, 5.
  of sound in air, dependence of, upon temperature, 244.
Vibration of air columns, 191.
  of diaphragms, 199.
  microscope, the, 185.
  of a particle, 177.
  of strings, 188.
Vibrations, forced, 201.
  free, 201.
  graphical representation of, 178.
  impressed, 201.
  longitudinal, of rods and strings, 197.
  proper, 201.
  simple and compound, 177.
  superposition of, 179.
  transverse, of rods and plates, 198.

VISION, dichroic, 136, 139.
  trichroic, 136.
Visual angle, 62.
Vocal organs, the, 196.
Vowel sounds, 204.

WAVE front, definition of, 21.
  theory of light and sound, 3.
  -length, definition of, 11.
  -length, change of, due to motion, 228.
  -length units, 120.
  -lengths, table of, 119.
  motion, equations of, 238.
  trains, 11.
  train, amplitude of, 13.
  train, compound, 12.
  train, intensity of, 13.
  train, phase of, at a point, 13.
  train, simple, 12.
  train, simple, equation of, 12.
  train, stationary, 14.
Waves from periodic disturbances, 11.
  longitudinal, 10.
  nature of, 10.
  superposition of, 14.
  transverse, 10.
White bodies, 110.
  light, definition of, 136.
Wide-angle lens, the, 84.
  -angle lenses, 82.
  -aperture lenses, 82.

ZONE plate, the, 109.
Zones, half-period, 106, 109.

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